Master Thesis

In fulfillment of the requirement for the degree of
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The proper base change formula in étale cohomology

Author:
Nicolai Palm
(nicolai.palm@stud.uni-regensburg.de)

Advisor:
Prof. Dr. Denis-Charles Cisinski

Faculty of mathematics
University of Regensburg

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0 Introduction

Within étale cohomology, the “proper base change theorem” can be considered to be of central importance. The theorem represents one of the special features within étale cohomology required for the proof of the “smooth base change” theorem or other high-level theorems. At the same time, elaboration of the “proper base change theorem” is either embedded within an extensive body of work [1] [2] or strongly reduced to core statements [3] with a hard to follow chain of argumentation. This paper aims to bridge the gap between these extremes. It is intended to provide access to understanding of the core chain of argumentation for all those readers who have a basic knowledge of a well-defined, limited number of mathematical areas, with emphasis on:

Algebraic geometry: it is assumed that readers are familiar with basic constructions and results within the theory of schemes, such as those presented in [4]. In addition, we strongly recommend familiarity with the theory of quasi-coherent sheaves and their cohomology. We not only use results from this theory, but also make use of its proof techniques. Good sources for further reading are [4] and [5].

Category theory: Quite often, we will use the language of locally pesentable categories, that allow us to elegantly derive central technical results. The theory as covered in [6] can be considered as sufficient basic knowledge. Alternatively, [7] may also serve as a fitting reference.

Homological algebra: We assume familiarity with the language of abelian categories and derived functors as well as with spectral sequences, in particular Grothendieck’s spectral sequence. A good reference for these elements may be found in [8].

The complete master thesis is structured as follows:

In chapter 1, we generalize the notion of a sheaf on a topological space. Thereby we observe, at the level of sheaves there is no difference between the category of open subsets of X and the category of local isomorphims over X (see theorem 1.9). This suggests definition of the class of local isomorphisms on a scheme in order to“refine” its underlying topology. We introduce the notion of a site which serves as a generalization of topological spaces in the sheaf sense. The notion of site comes with a canonical notion of a sheaf. We use the theory of locally presentable categories to prove the category of sheaves to satisfie good properties. In particular, the category of abelian group objects of sheaves is an abelian Grothendieck category so we can apply the machinerie of derived functors. We then define sheaf cohomology to be the derived functor of some section functor. Results within this context are well known for ordinary sheaf theory. Related proof sequences, accordingly, are only sketched or referenced in order to prevent overloading this chapter.

In chapter 2, we define étale morphisms of schemes. They mimic the notation of local isomorphisms within complex analytic topology. As this theory is primarily rooted in well elaborated commutative algebra, we only state later on required properties. [5] may serve as a comprising source for readers with further interest in details. We define the étale site on a scheme X as the category of étale morphisms together with jointly surjective families as coverings. And we define étale sheaves as sheaves on the étale site and étale cohomology as sheaf cohomology of the global section functor. We show that some étale presheaves satisfie the sheaf condition, including representable presheaves and a suitible generalization of the structure sheaf of some scheme X to the étale site. We will point out that étale cohomology generalizes cohomology of quasi-coherent sheaves. By using the theory base of chapter 1, then, allows us to elaborate theorems specifically valid within the category of étale sheaves to provide a solid theory for étale cohomology. The main focus lies on the compatibility between étale cohomology and cofiltered limits of schemes. This general framework provides a main technic within the proof of the proper base change theorem.

In chapter 3, we introduce the notion of étale neighbourhoods and stalks. Then, we explore the algebraic properties of the étale stalk of the generalized structure sheaf. The étale stalk at a point, thereby, turns out to be the strict henselianization of the Zariski stalk of the structure sheaf at that point. We use our so far gained insights and results to prove the direct image of a finite morphism of schemes to be exact. This is a useful property of the étale site, while not being true for the Zariski topology.

In chapter 4, we introduce the Artin approximation theorem (see theorem 4.4). For suitable cases, it reveals henselian local rings to contain similar data as their adic completions.

In chapter 5, we explore properties of the first and second étale cohomology group. We prove the first étale cohomology group to classiy torsors just as the topological sheaf cohomology group does. Furthermore, we will point out that étale cohomology generalizes Galois cohomology. Both types - Galois cohomology as well as cohomology of quasi-coherent sheaves - reveal good properties indicating the same for étale cohomology. We apply our so far gained results on the case of curves, i.e. for schemes of finite type over some seperably closed field k and of dimension less or equal one. As a result, we may relate the cohomology groups of curves to the cohomology groups of points being finite or of transcendence degree 1 over k.

Chapter 6 concludes the proof sequence by putting together elements of previous chapters. The chapter starts with the introduction of a generalized framework for base changing based on own considerations. Known morphisms, then, are identified as instances of this generalized framework. At this point, finally, the proper base change theorem may be phrased. We then sketch an overview of its proof sequence by referencing to the framework as prepared in previous chapters. This summary together with the content of previous chapters will provide thorough understanding for readers in two directions: Keep track of the proof's sequence while, simultaneously, allowing to dive into its individual constituents.

Mastering the proper base change theorem proof represents a tour de force. The author himself got lost during several attempts. A summary, as developed at a later stage of the work helped clear the fog and keep the trail along a consistent path without unnecessary detours. In addition, the general framework for base changing helped to understand many of the arguments within the proof. At this level of abstraction, the proper base change proof sequence becomes a clear undertaking. It is my strong hope that this will be of great help not only to me as author but also to all interested readers.

1 Sheaves and cohomology

All rings apearing in this thesis are assumed to be commutative and unital.

1.1 Sites

Definition 1.1. Let $\mathcal{C}$ be a category. A *family of morphisms (with fixed target U)* denoted by

$$\{\phi_i : U_i \to U\}_{i\in I}$$

is the data of a set I together with morphisms $\phi_i : U_i \to U$ in $\mathcal{C}$ for all i in I. A morphism

$$\{\phi_i : U_i \to U\}_{i\in I} \to \{\psi_j : V_j \to V\}_{j\in J}$$

is given by a map of sets $a : I \to J$, morphisms $f_i : U_i \to V_{a(i)}$ for all $i \in I$ and $f : U \to V$ such that

$$\begin{tikzcd}
U_i \arrow[r, "f_i"] \arrow[d, "\phi_i"] & V_{a(i)} \arrow[d, "\psi_{a(i)}"] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

commutes. A *refinement* of $\{\phi_i : U_i \to U\}_{i\in I}$ is given by a morphism

$$\{\phi_i : U_i \to U\}_{i\in I} \to \{\psi_j : V_j \to U\}_{j\in J}$$

such that $f = id_U$.

The notion of a site is what we need in order to make the notion of a sheaf on a small category reasonable.

Definition 1.2 (Site)**.** A *site* T is a tupel consisting of a small category $\mathcal{A}$, whose objects are called opens, together with a set $\mathrm{Cov}(\mathcal{A})$ of families $\{\phi_i : U_i \to U\}_{i\in I}$, called coverings of $\mathcal{A}$ satisfying the following axioms:

1. Given a covering $\{\phi_i : U_i \to U\}_{i\in I}$ all fibre products $U_i \times_U U_j$ exist and the induced family $\{U_i \times_U U_j \to U_j\}_{i\in I}$ is also a covering.
2. Given a covering $\{\phi_i : U_i \to U\}_{i\in I}$ and coverings $\{\phi_{ij} : U_{ij} \to U_i\}_{j\in J_i}$ for each $i \in I$ the induced family $\{\phi_i \circ \phi_{ij} : U_{ij} \to U\}_{i\in I, j\in J_i}$ is a covering.
3. Given an isomorphism $U \xrightarrow{\cong} V$ in $\mathcal{A}$, then, $\{U \xrightarrow{\cong} V\}$ is a covering.

A morphism of sites $(\mathcal{A}, \mathrm{Cov}(\mathcal{A})) \to (\mathcal{A}', \mathrm{Cov}(\mathcal{A}'))$ is given by a functor $F : \mathcal{A} \to \mathcal{A}'$ which preserves all coverings, i.e. given a covering $\{\phi_i : U_i \to U\}_{i\in I}$ in $\mathcal{A}$, then,

$$\{F(\phi_i) : F(U_i) \to F(U)\}_{i\in I}$$

is a covering in $\mathcal{A}'$, and preserves all fibre products appearing in 1. of the above definition.

Example 1.3. Let X be a topological space. Denote by $\mathrm{Ouv}(X)$ the category of open subsets of X together with morphisms the canonical inclusions. Define a covering in $\mathrm{Ouv}(X)$ to be a family $\{U_i \subset U\}_{i\in I}$ such that $\cup_{i\in I} U_i = U$. We can check this to define a site which we also denote by $\mathrm{Ouv}(X)$.

Example 1.4. Recall a continuous map $X \to Y$ of topological spaces to be a local homeomorphism if for every $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that the restriction $f : U \to f(U)$ is an isomorphism of topological spaces.

Let X be a topological space. Denote by $\mathrm{Et}(X)$ the full subcategory of topological spaces over X consisting of local homeomorphisms. By declaring coverings to be jointly surjective families of morphisms, we obtain a site denoted by $\mathrm{Et}(X)$. Every inclusion of an open subset of X is a local homeomorphism. Thus, the canonical functor

$$\mathrm{Ouv}(X) \to Et(X), U \mapsto U$$

induces a morphism of sites.

1.2 Sheaves on a site

For the rest of this section we denote by T a site and by $\mathcal{C}$ a category which has all limits which appear. As a convention, we will treat T and its underlying category as the same whenever it is clear from context what we mean.

1.2.1 Definition and examples

Definition 1.5 (Sheaves on a site)**.** A presheaf $\mathcal{F} \in \mathrm{PSh}(T, \mathcal{C})$ *satisfies the sheaf condition for a covering* $\{\phi_i : U_i \to U\}_{i\in I}$ if the diagram

$$\mathcal{F}(U) \overset{\alpha}{\to} \prod_I \mathcal{F}(U_i) \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \prod_{I^2} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram where α is induced by the family $\mathcal{F}(\phi_i)$, α_1 by the first projection $U_i \times_U U_j \to U_i$ and α_2 by the second projection $U_i \times_U U_j \to U_j$. We say $\mathcal{F}$ is a *sheaf on* T *with values in* $\mathcal{C}$ if it satisfies the sheaf condition for all coverings in T. If $\mathcal{C} = \mathrm{Ab}$ we say $\mathcal{F}$ is an abelian sheaf. The full subcategory of $\mathrm{PSh}(\mathcal{A}, \mathcal{C})$ consisting of sheaves is denoted by $\mathrm{Sh}(T, \mathcal{C})$.

Example 1.6. Let X be a topological space. Then, the category of sheaves on $\mathrm{Ouv}(X)$ agrees with the category of sheaves on X defined in the usual way.

Definition 1.7 (Zariski site)**.** Let X be a scheme. We define the *Zariski site* denoted by Zar_X to be $\mathrm{Ouv}(X)$ of the underlying topological space of X defined in example 1.3.

Example 1.8. Let X be a topological space. Every X-space Y gives rise to a sheaf on X given by

$$\mathrm{Hom}_X(-,Y) : \mathrm{Top}/X \to \mathrm{Set}$$

restricted to $\mathrm{Et}(X)$.

Here is a crucial theorem.

Theorem 1.9. *Let X be a topological space. Restricting a sheaf of sets from* $\mathrm{Et}(X)$ *to* $\mathrm{Ouv}(X)$ *induces an equivalence of categories*

$$\mathrm{Sh}(X, \mathrm{Set}) \simeq \mathrm{Sh}(Et(X), \mathrm{Set}).$$

Proof. II.6 Corollary 3. in [9]. □

1.2.2 Čech complex

In ordinary sheaf cohomology, the Čech-to-derived functor spectral sequence is a useful tool in order to study sheaf cohomology groups. We can extend the notion and results in Čech-cohomology on topological spaces canonically to sheaves on sites.

Definition 1.10 (Čech complex)**.** Let $\{U_i \to U\}_{i\in I}$ be a family of morphisms in T and $\mathcal{F} \in \mathrm{PSh}(T, \mathrm{Ab})$ be an abelian presheaf. Assume all appearing fibre products to exist. We define

$$\check{C}^q(\{U_i \to U\}_{i\in I}, \mathcal{F}) = \prod_{(i_0,\dots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_q}) \in \mathrm{Ab}$$

and morphisms $d_q : \check{C}^q(\{U_i \to U\}_{i\in I}, \mathcal{F}) \to \check{C}^{q+1}(\{U_i \to U\}_{i\in I}, \mathcal{F})$ induced by

$$d_q((s_{(i_0,\dots,i_q)})_{I^{q+1}}) = \sum_{j=0}^{q+1} (-1)^j s_{(i_0,\dots,\hat{i}_j,\dots,i_q)}|_{U_{i_0}\times_U \cdots \times_U U_{i_q}} \in \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_q})$$

where the restriction is induced by the projection

$$U_{i_0} \times_U \cdots \times_U U_{i_q} \to U_{i_0} \times_U \cdots \times_U \hat{U}_{i_j} \times_U \cdots \times_U U_{i_q}.$$

Proving $d_{q+1} \circ d_q = 0$ is straightforward. We obtain a complex

$$\check{C}^\bullet(\{U_i \to U\}_{i\in I}, \mathcal{F}) = (\check{C}^0(\{U_i \to U\}_{i\in I}, \mathcal{F}) \overset{d_0}{\to} \check{C}^1(\{U_i \to U\}_{i\in I}, \mathcal{F}) \overset{d_1}{\to} \cdots)$$

called *Čech complex*. We define

$$\check{H}^q(\{U_i \to U\}_{i\in I}, \mathcal{F}) = \mathrm{H}^q(\check{C}^\bullet(\{U_i \to U\}_{i\in I}, \mathcal{F}))$$

to be the *q-th Čech cohomology group with respect to* $\{U_i \to U\}_{i\in I}$.

Theorem 1.11. *Let T be a site and $\mathcal{F}$ an abelian presheaf on T. Then,*

1. *$\mathcal{F}$ is a sheaf iff for every covering $\{U_i \to U\}_{i\in I}$ in T the canonical morphism*

$$\mathcal{F}(U) \to \check{H}^0(\{U_i \to U\}_{i\in I}, \mathcal{F})$$

is an isomorphism.

2. *the assignement $\mathcal{F} \mapsto \check{C}^\bullet(\{U_i \to U\}_{i\in I}, \mathcal{F})$ extends canonically to an exact functor*

$$\mathrm{PSh}(T, \mathrm{Ab}) \to \mathrm{Comp}^+(\mathrm{Ab})$$

of abelian categories.

3. *there exists a quasi-isomorphism*

$$\check{C}^\bullet(\{U_i \to U\}_{i\in I}, \mathcal{F}) \to R\check{H}^0(\{U_i \to U\}_{i\in I}, \mathcal{F})$$

functorial in $\mathcal{F}$. In particular, we obtain an isomorphism

$$\check{H}^q(\{U_i \to U\}_{i\in I}, -) \cong R^q\check{H}^0(\{U_i \to U\}_{i\in I}, -)$$

of functors for every q.

Proof. 1. [10, Tag 03AN]

2. [10, Tag 03AQ]

3. [10, Tag 03AU]

□

Definition 1.12. Let T be a site and $U \in T$. We define the *category of coverings of U* denoted by $\mathrm{Cov}(U)$ to have objects all covering families of U and morphisms as in definition 1.1 with canonical composition and identity.

Remark 1.13. Assume T has all fibre products. Then, $\mathrm{Cov}(U)^{op}$ is a filtered category. The assignement $\{U_i \to U\}_{i\in I} \mapsto \check{C}^\bullet(\{U_i \to U\}_{i\in I}, \mathcal{F})$ extends canonically to a functor

$$\check{C}^\bullet(-, \mathcal{F}) : \mathrm{Cov}(U)^{op} \to \mathrm{Comp}^+(\mathrm{Ab}).$$

We define $\check{C}^\bullet(U, \mathcal{F})$ to be its colimit in $\mathrm{Comp}^+(\mathrm{Ab})$ and $\check{H}^q(U, \mathcal{F})$ to be its q-th cohomology group. Furthermore, we can check the assignement $\mathcal{F} \mapsto \check{C}^\bullet(U, \mathcal{F})$ to extend to a functor

$$\check{C}^\bullet(U, -) : \mathrm{PSh}(T, \mathrm{Ab}) \to \mathrm{Comp}^+(\mathrm{Ab}).$$

Observe $\check{H}^0(U, -)$ to be left exact since $\mathrm{Cov}(U)^{op}$ is filtered and by 2. of theorem 1.11. We deduce the functorial quasi-isomorphism in 3. of theorem 1.11 to extend to a functorial quasi-isomorphism $\check{C}^\bullet(U, \mathcal{F}) \to R\check{H}^0(U, \mathcal{F})$ using $\mathrm{Cov}(U)^{op}$ to be filtered and filtered colimits to be exact.

1.2.3 Categorical properties of $\mathrm{Sh}(T, \mathcal{C})$

Assume $\mathcal{C}$ is either $\mathrm{Set}, \mathrm{Ab}$ or ${}_\Lambda\mathrm{Mod}$ with Λ some ring. For simplicity we will not distinguish between $\mathcal{F} \in \mathrm{Sh}(T, \mathcal{C})$ and its image under the inclusion in $\mathrm{PSh}(T, \mathcal{C})$ whenever it is not necessary.

Remark 1.14. Recall the category of presheaves $\mathrm{PSh}(T, \mathcal{C})$ to be locally finitely presentable and colimits as well as limits to be computed pointwise. In addition, a strong generator of finitely presentable objects in $\mathrm{PSh}(T, \mathrm{Set})$ is given by representable presheaves. Furthermore, let

$$\Lambda[-] : \mathrm{Set} \to {}_\Lambda\mathrm{Mod}$$

be the left adjoint to the forgetful functor. We obtain a left adjoint

$$\mathrm{PSh}(T, \mathrm{Set}) \to \mathrm{PSh}(T, {}_\Lambda\mathrm{Mod}), \mathcal{F} \mapsto \Lambda[-] \circ \mathcal{F}$$

of the forgetful functor.

Remark 1.15. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Recall the following notation. We say F commutes with κ-filtered colimits if the induced $(F(D(i)) \to F(A))_I$ is a colimit cocone for every κ-filtered diagram $D : I \to \mathcal{A}$ and $(D(i) \to A)_I$ a colimit cocone in $\mathcal{A}$. We say κ-filtered colimits commute with κ-finite limits in $\mathcal{A}$ if for all diagrams $D : I \times J \longrightarrow \mathcal{A}$ with I κ-filtered and J κ-finite the induced morphism

$$\operatorname*{colim}_{i \in I}(\lim_{j \in J}(F(i,j))) \longrightarrow \lim_{j \in J}(\operatorname*{colim}_{i \in I}(F(i,j)))$$

is an isomorphism whenever both, limit and colimit, exist.

Lemma 1.16. *Let $\mathcal{F} \in \mathrm{PSh}(T, \mathrm{Set})$ be a presheaf and $\{U_i \to U\}_{i \in I}$ be a covering in T. Define*

$$\check{H}^0(\{U_i \to U\}_{i \in I}, \mathcal{F}) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) |\ s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \forall i, j \in I\}.$$

This asssignemnt extends to a functor

$$\check{H}^0(-,\mathcal{F}) : \mathrm{Cov}(U)^{op} \to \mathrm{Set}, \{U_i \to U\}_{i\in I} \mapsto \check{H}^0(\{U_i \to U\}_{i\in I}, \mathcal{F})$$

for every $U \in T$. We define $\mathcal{F}^+(U)$ to be a colimit of this diagram.

Furthermore, basechanging induces a functor

$$\mathrm{Cov}(V) \to \mathrm{Cov}(U), (V_i \to V) \mapsto (V_i \times_V U \to U)$$

for every $U \to V$ in T and, hence, a unique morphism $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. We obtain a presheaf $\mathcal{F}^+ \in \mathrm{PSh}(T, \mathrm{Set})$ together with a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^+$ induced by the family $\mathcal{F}(U) \to \mathcal{F}(U_i)$.

Proof. [10, Tag 00W4] □

Theorem 1.17. *The canonical inclusion*

$$\mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$$

admits an exact left adjoint $(-)^{\#} = ((-)^+)^+$ called sheafification*. Furthermore, let $\mathcal{F} \in \mathrm{PSh}(T, \mathrm{Set})$ be a presheaf, then, its unit is given by the composition*

$$\mathcal{F} \to \mathcal{F}^+ \to (\mathcal{F}^+)^+$$

of the canonical morphisms of the previous lemma. In addition, the canonical morphism

$$\mathcal{F}^+ \to (\mathcal{F}^+)^+$$

is injective.

Proof. For the first part for sheaves of sets see [10, Tag 00WH] and [10, Tag 00WJ] For the second part see [10, Tag 00WB] In general, since $(-)^{\#}$ preserves finite limits it preserves algebraic structures. Therefore, we obtain a unique factorization

$$\begin{tikzcd}
\mathrm{PSh}(T, \mathcal{C}) \arrow[r, dashed, "(-)^{\#}"] \arrow[d, "forget"] & \mathrm{Sh}(T, \mathcal{C}) \arrow[d, "forget"] \\
\mathrm{PSh}(T, \mathrm{Set}) \arrow[r, "(-)^{\#}"] & \mathrm{Sh}(T, \mathrm{Set})
\end{tikzcd}$$

for $\mathcal{C} = \mathrm{Ab}, {}_{\Lambda}\mathrm{Mod}$ which is an exact left adjoint to the inclusion $\mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$. □

Remark 1.18. Since $\mathrm{Sh}(T,\mathcal{C}) \subset \mathrm{PSh}(T,\mathcal{C})$ is reflective, the category of sheaves is complete and cocomplete by Lemma 2.30 and Lemma 2.31 in [6]. We recall the explicit constructions of limits resp. colimits of sheaves. Let $D : I \to \mathrm{Sh}(T,\mathcal{C})$ be a diagram. Let $(\mathcal{F} \to \iota D(i))_I$ be a limit diagram resp. $(\iota D(i) \to \mathcal{F})_I$ be a colimit diagram in $\mathrm{PSh}(T,\mathcal{C})$. Then, the cone

$$(\mathcal{F}^\# \to \iota D(i))_I$$

is a limit diagram resp. the cocone

$$(\iota D(i) \to \mathcal{F} \to \mathcal{F}^\#)_I$$

is a colimit cocone in $\mathrm{Sh}(T,\mathcal{C})$ for $\mathcal{F}^\# \to D(i)$ the unique extension of $\mathcal{F} \to D(i)$ along $\mathcal{F} \to \mathcal{F}^\#$ and $\mathcal{F} \to \mathcal{F}^\#$ the unit.

Recall the following result of locally presentable categories.

Lemma 1.19. *Let $\mathcal{B}$ be a locally κ-presentable category. Assume $\mathcal{A} \subset \mathcal{B}$ to be a reflective subcategorie and the inclusion to preserve κ-filtered colimits Then, $\mathcal{B}$ is locally κ-presentable and a strong generator of κ-presentable objects is given by the images of a strong generator under the reflector.*

Proof. Lemma 2.32 in [6]. □

Since the site T is (essentially) small we may find a regular cardinal bounding the size of all covering families. This will be an upper bound for the local presentability of $\mathrm{Sh}(T,\mathcal{C})$.

Corollary 1.20. *Let κ be a regular cardinal such that for all coverings $\{U_i \to U\}_{i\in I}$ in T the cardinality of I is smaller than κ. Then, the category $\mathrm{Sh}(T,\mathcal{C})$ is locally κ-presentable. In addition, for $\mathcal{C} = \mathrm{Set}$, a strong generator is given by the set of sheafifications of representable presheaves.*

Proof. It suffices to prove the inclusion $\mathrm{Sh}(T,\mathcal{C}) \subset \mathrm{PSh}(T,\mathcal{C})$ to preserve κ-filtered colimits by the previous lemma. Let

$$(\mathcal{F}_d \to \mathcal{F})_D$$

be a κ-filtered colimit cocone in $\mathrm{Sh}(T,\mathcal{C})$ and

$$(\mathcal{F}_d \to \mathcal{G})_D$$

be a colimit cocone in $\mathrm{PSh}(T,\mathcal{C})$. Then, $\mathcal{G}^\# \cong \mathcal{F}$ are canonically isomorphic by remark 1.18. Therefore, it suffices to prove that $\mathcal{G}$ is a sheaf. Let $\{U_i \to U\}_{i\in I}$ be a covering in T. By assumption, I is κ-finite. Observe $\mathrm{PSh}(T, \mathcal{C})$ to be κ-presentable since it is finitely presentable. We deduce

$$\mathrm{equ}(\prod_I \mathcal{G}(U_i) \rightrightarrows \prod_{I^2} \mathcal{G}(U_i \times_U U_j)) \cong \mathrm{colim}_D \, \mathrm{equ}(\prod_I \mathcal{F}_d(U_i) \rightrightarrows \prod_{I^2} \mathcal{F}_d(U_i \times_U U_j)) \cong \mathcal{G}(U)$$

to be canonically isomorphic since κ-filtered colimits commute with κ-finite limits in $\mathrm{PSh}(T, \mathcal{C})$. □

Lemma 1.21. *Finite limits commute with filtered colimits in* $\mathrm{Sh}(T, \mathcal{C})$*.*

Proof. Let $F : I \times J \longrightarrow \mathrm{Sh}(T, \mathcal{C})$ be a diagram with I filtered and J finite. Denote by

$$\iota : \mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$$

the inclusion. Since $\mathrm{PSh}(T, \mathcal{C})$ is locally finitely presentable the induced

$$\operatorname*{colim}_{i \in I}(\lim_{j \in J}(\iota F(i,j))) \longrightarrow \lim_{j \in J}(\operatorname*{colim}_{i \in I}(\iota F(i,j)))$$

is an isomorphism in $\mathrm{PSh}(T, \mathcal{C})$. Then, the induced

$$\operatorname*{colim}_{i \in I}(\lim_{j \in J}((-)^{\#} \circ \iota F(i,j))) \longrightarrow \lim_{j \in J}(\operatorname*{colim}_{i \in I}((-)^{\#} \circ \iota F(i,j))$$

is an isomorphism in $\mathrm{Sh}(T, \mathcal{C})$ since $(-)^{\#}$ preserves finite limits. Since the counit of $(-)^{\#} \dashv \iota$ is an isomorphism we deduce the claim. □

Remark 1.22. There are several different constructions of the sheafification functor. For example, we could have noticed $\mathcal{F} \in \mathrm{PSh}(T, \mathcal{C})$ to be a sheaf iff the canonical morphisms

$$K = \mathrm{coker}(\sqcup_{I^2} h_{U_i \times_U U_j} \rightrightarrows \sqcup_I h_{U_i}) \to h_U$$

induce isomorphisms

$$\mathrm{Hom}_{\mathrm{PSh}(T,\mathcal{C})}(h_U, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{PSh}(T,\mathcal{C})}(K, \mathcal{F})$$

for every covering $\{U_i \to U\}_{i \in I}$ in T. Such subcategories defined by a set of arrows in a locally presentable category are always reflective by Theorem 3.3 and Remark 3.19 in [6] However, this approach does not easily implie the exactness of $(-)^{\#}$.

We can in addition prove the following lemma using standard arguments.

Lemma 1.23. *If $\mathcal{C} = \mathrm{Ab}, {}_\Lambda\mathrm{Mod}$, then, the abelian category strucure on $\mathrm{PSh}(T, \mathcal{C})$ induces canonically an abelian category structure on $\mathrm{Sh}(T, \mathcal{C})$ such that the inclusion $\mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$ is additive.*

Corollary 1.24. *If $\mathcal{C} = \mathrm{Ab}, {}_\Lambda\mathrm{Mod}$, then, $\mathrm{Sh}(T, \mathcal{C})$ is abelian and has enough injective objects.*

Proof. It suffices to prove $\mathrm{Sh}(T, \mathcal{C})$ to be abelian, locally presentable and filtered colimits to commute with finite limits by Lemma 4.11 in [6] using lemma 1.21 and corollary 1.20. This is lemma 1.23, lemma 1.21 and corollary 1.20. ☐

We ask under which conditions $\mathrm{Sh}(T, \mathcal{C})$ is locally finitely presentable.

Definition 1.25. The site T is *noetherian* if every covering $\{U_i \to U\}_{i \in I}$ admits some $J \subset I$ finite and a refinement of $\{U_i \to U\}_{i \in I}$ by $\{U_j \to U\}_{j \in J}$.

Lemma 1.26. *Assume T to be noetherian. Then, filtered colimits of sheaves in $\mathrm{Sh}(T, \mathcal{C})$ are computed pointwise, i.e. the inclusion*

$$\mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$$

commutes with filtered colimits.

Proof. Let $(\mathcal{F}_i \to \mathcal{F})_I$ be a colimit cocone in $\mathrm{Sh}(T, \mathcal{C})$ and $(\mathcal{F}_i \to \mathcal{G})_I$ be a colimit cocone in $\mathrm{PSh}(T, \mathcal{C})$. The induced $\mathcal{G}^\# \to \mathcal{F}$ is an isomorphism by remark 1.18. Therefore, it suffices to prove that $\mathcal{G}$ is a sheaf. We can reduce to proving that $\mathcal{G}$ satisfies the sheaf condition for all finite coverings since we can refine every covering by a finite covering. We deduce the claim by a similar arguments as in corollary 1.20. ☐

Corollary 1.27. *Assume T is noetherian. Then, $\mathrm{Sh}(T, \mathcal{C})$ is locally finitely presentable. If $\mathcal{C} = \mathrm{Set}$, the set of sheafifications of representable presheaves is a strong generator of finitely presentable objects.*

Proof. This is essentially remark 1.14, theorem 1.17, lemma 1.19 and the previous lemma. ☐

1.3 Cohomology of sheaves

We assume in addition $\mathcal{C} = \mathrm{Ab}, {}_\Lambda\mathrm{Mod}$ for some ring Λ for the rest of this section. Denote by $\iota : \mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$ the canonical inclusion.

Definition 1.28. Let U be in T. Denote by

$$\Gamma^{\mathrm{P}}(\mathrm{U}, \text{-}) : \mathrm{PSh}(T, \mathcal{C}) \longrightarrow \mathcal{C}, \mathcal{F} \mapsto \mathcal{F}(U), f \mapsto \mathcal{F}(f)$$

the functor evaluating a presheaf at U and by

$$\Gamma(\mathrm{U},\text{-}) = \Gamma^{\mathrm{P}}(\mathrm{U},\text{-}) \circ \iota : \mathrm{Sh}(T,\mathcal{C}) \longrightarrow \mathcal{C}$$

its restriction to $\mathrm{Sh}(T,\mathcal{C})$. Then, $\Gamma(\mathrm{U},\text{-})$ is left exact since $\Gamma^{\mathrm{P}}(\mathrm{U},\text{-})$ is exact and ι is a right adjoint by theorem 1.17. Therefore, the right derived functor of $\Gamma(\mathrm{U},\text{-})$ exists since $\mathrm{Sh}(T,\mathcal{C})$ has enough injective objects by corollary 1.24. Let $\mathcal{F} \in \mathrm{Sh}(T,\mathcal{C})$ be a sheaf. We define the *p-th sheaf cohomology group of U with values in $\mathcal{F}$*

$$\mathrm{H}^p(U,\mathcal{F}) = R^p\,\Gamma(\mathrm{U},\text{-})(\mathcal{F}) = \mathrm{H}^p(R\,\Gamma(\mathrm{U},\text{-})(\mathcal{F}))$$

to be the the p-th right-derived functor of $\Gamma(\mathrm{U},\text{-})$ for every $p \geq 0$.

Definition 1.29. Let $f : \mathrm{T} \to \mathrm{T}'$ be a morphism of sites. By abstract reasons, namely by the existence of a left and right kan-extension, the functor

$$f_*^P : \mathrm{PSh}(\mathrm{T}',\mathcal{C}) \to \mathrm{PSh}(\mathrm{T},\mathcal{C}), \mathcal{F} \mapsto (U \mapsto \mathcal{F}(f(U)))$$

has a right and a left adjoint. We denote its left adjoint by f_P^*. Then, f_*^P restricts to a functor

$$f_* : \mathrm{Sh}(\mathrm{T}',\mathrm{Set}) \to \mathrm{Sh}(\mathrm{T},\mathrm{Set}), \mathcal{F} \mapsto (U \mapsto \mathcal{F}(f(U)))$$

since f preserves coverings. Sheafifying f_P^*, we obtain a left adjoint f^* of f_* given by

$$(-)^{\#} \circ f_P^* : \mathrm{Sh}(\mathrm{T},\mathrm{Set}) \to \mathrm{Sh}(\mathrm{T}',\mathrm{Set}), \mathcal{F} \mapsto (U \mapsto \operatorname*{colim}_{V \in f/U} \mathcal{F}(V))^{\#}.$$

We observe f/U to be cofiltered if T has fibre products and f preserves them. In that case f^* is exact.

Remark 1.30. Let $f : \mathrm{T} \to \mathrm{T}'$ be a morphism of sites. By the very construction of the inverse and direct image we observe the unit $\epsilon : 1 \Rightarrow f_* f^*$ to be pointwise given by

$$\epsilon_{\mathcal{F}}(U) : \mathcal{F}(U) \to \operatorname*{colim}_{f/f(U)} \mathcal{F}(V) = f_*^P f_P^*\,\mathcal{F}(U) \to f^*\,\mathcal{F}(f(U))$$

where the first map is the colimit cocone morphism corresponding to $id_{f(U)}$ and the second morphism is induced by the unit $f_P^*\,\mathcal{F} \to (f_P^*\,\mathcal{F})^{\#} = f^*\,\mathcal{F}$.

Example 1.31. Let $*$ denote the category with a single object pt and the identity. Let $X \in T$ be an object in T. Then, $*$ is canonical a site and

$$i : * \to T, pt \mapsto X$$

is a morphism of sites. We obtain a (up to natural isomorphism) commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(T, \mathcal{C}) \arrow[r, "i_*"] \arrow[d, "\Gamma(\mathrm{X,\text{-}})"] & \mathrm{Sh}(*, \mathcal{C}) \arrow[dl, "\Gamma(\mathrm{pt,\text{-}})"] \\
\mathcal{C} &
\end{tikzcd} .$$

We can check that $\Gamma(\mathrm{pt}, \text{-})$ is an equivalence. In particular, $R^q i_*$ computes sheaf cohomology of X and $\Gamma(\mathrm{X}, \text{-})$ admits a left-adjoint.

Lemma 1.32. *Let $L : \mathcal{A} \to \mathcal{B}$ be a left adjoint and $\mathcal{A}, \mathcal{B}$ be abelian categories. If L is exact, then, its right adjoint R preserves injective objects.*

Proof. Let $I \in \mathcal{B}$ be an injective object. Then,

$$\mathrm{Hom}(-, R(I)) \cong \mathrm{Hom}(L(-), I)$$

is exact since it is a composition of exact functors. □

Corollary 1.33. *The inclusion $\iota : \mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$ preserves injective objects. Furthermore, given some morphism of sites $f : T \to T'$, the functor*

$$f^* : \mathrm{Sh}(T, \mathcal{C}) \to \mathrm{Sh}(T', \mathcal{C})$$

preserves injective objects if T has fibre products and f preserves them.

Proof. It suffices to prove their respective left adjoints to be exact by the previous lemma. For ι this is theorem 1.17. For f_* this was observed in definition 1.29. □

Lemma 1.34. *Let $f : T \to T'$ be a morphism of sites. Then, the q-th right derived functor $R^q f_* \mathcal{F}$ of f_* is natural isomorphic to the sheafification of the presheaf*

$$U \mapsto \mathrm{H}^q(f(U), \mathcal{F}).$$

Proof. Let $0 \to \mathcal{F} \to I^\bullet$ be an injective resolution of $\mathcal{F}$. Then, $R^q f_* \mathcal{F} \cong \mathrm{H}^q(f_* I^\bullet)$ are isomorphic. Thus, we deduce an isomorphism $\mathrm{H}^q(f_* I^\bullet) \cong (U \mapsto \mathrm{H}^q(f_* I^\bullet(U)))^{\#}$ by remark 1.18. At last, we observe

$$\mathrm{H}^q(f_* I^\bullet(U)) = \mathrm{H}^q(I^\bullet(f(U))) \cong \mathrm{H}^q(f(U), \mathcal{F})$$

to be isomorphic. Combined we deduce the claim. □

1.4 Calculation of sheaf cohomology

1.4.1 Comparison spectral sequences

Remark 1.35. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories and

$$0 \to A \to A_0 \to A_1 \to \cdots$$

be a left-bounded complex of objects in $\mathcal{A}$. Assume its right derived functor

$$RF : D^+ \mathcal{A} \to D^+ \mathcal{B}$$

to exist. We say

$$0 \to A \to A_0 \to A_1 \to \cdots$$

computes RF if the induced

$$RF(A) \to (FA_0 \to FA_1 \to \cdots)$$

is an isomorphism in $D^+ \mathcal{B}$.

Let us recall the notion of an adapted class and its relation to derived functors.

Remark 1.36. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian category. Recall a class of objects $\mathcal{R}$ in $\mathcal{A}$ to be adapted to F if

1. $\mathcal{R}$ is stable under taking finite direct sums.
2. F maps acyclic complexes of objects in $\mathcal{R}$ to acyclic complexes.
3. Any object in $\mathcal{A}$ is a subobject of an object in $\mathcal{R}$.

If $\mathcal{R}$ is adapted to F, then, for every quasi-isomorphism $K^\bullet \to A^\bullet$ in $\mathrm{Comp}^+(\mathcal{A})$ with every object of $A^\bullet$ in $\mathcal{R}$ the induced $RF(K^\bullet) \to FA^\bullet$ is an isomorphism using III.6.8 Theorem in [8].

Remark 1.37. Let $\{U_i \to U\}$ be a covering in T. The following diagrams

$$\begin{tikzcd}
\mathrm{Sh}(T,\mathcal{C}) \arrow[r, "\Gamma(\mathrm{U,\text{-}})"] \arrow[d, "\iota"] & \mathcal{C} \\
\mathrm{PSh}(T,\mathcal{C}) \arrow[ur, "\Gamma^{\mathrm{P}}(\mathrm{U,\text{-}})"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sh}(T,\mathcal{C}) \arrow[r, "\Gamma(\mathrm{U,\text{-}})"] \arrow[d, "\iota"] & \mathcal{C} \\
\mathrm{PSh}(T,\mathcal{C}) \arrow[ur, "\check{H}^0(\{U_i \to U\}{,}-)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sh}(T,\mathcal{C}) \arrow[r, "\Gamma(\mathrm{U,\text{-}})"] \arrow[d, "\iota"] & \mathcal{C} \\
\mathrm{PSh}(T,\mathcal{C}) \arrow[ur, "\check{H}^0(U{,}-)"'] &
\end{tikzcd}$$

commute up to isomorphism by definition. Notice that we implicitely assumed $\mathcal{C} = \mathrm{Ab}$ whenever we use Čech cohomology.

Using the theory of spectral sequences, in particular Grothendieck's spectral sequence, we can compare sheaf cohomology to other derived functors.

Theorem 1.38. *Let $\{U_i \to U\}_{i\in I}$ be a covering in T and $\mathcal{F} \in \mathrm{Sh}(T,\mathcal{C})$. There exist converging spectral sequences*

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\{U_i \to U\},-)(R^q\iota(\mathcal{F})) \Rightarrow \mathrm{E}^{p+q} = \mathrm{H}^{p+q}(U,\mathcal{F})$$

$$\mathrm{E}_2^{pq} = \check{H}^p(U,-)(R^q\iota(\mathcal{F})) \Rightarrow \mathrm{E}^{p+q} = \mathrm{H}^{p+q}(U,\mathcal{F})$$

$$\mathrm{E}_2^{pq} = R^p\,\Gamma^{\mathrm{P}}(\mathrm{U},\text{-})(R^q\iota(\mathcal{F})) \Rightarrow \mathrm{E}^{p+q} = \mathrm{H}^{p+q}(U,\mathcal{F})$$

which are functorial in $\mathcal{F}$.

Proof. Recall the abelian categories $\mathrm{Sh}(T,\mathcal{C})$ and $\mathrm{PSh}(T,\mathcal{C})$ to have enough injective objects and ι to preserve injective objects and injective objects to form an adapted class to any left exact functor. Therefore, we can apply Grothendieck's spectral sequence to all three compositions in the previous remark in order to calculate $\mathrm{H}^p(U,\mathcal{F})$. □

Proposition 1.39. *Let U be an element of T and $\mathcal{F} \in \mathrm{Sh}(T,\mathrm{Ab})$ be a sheaf. Then,*

$$R^p\iota(\mathcal{F})(U) \cong \mathrm{H}^p(U,\mathcal{F})$$

are natural isomorphic.

Proof. The right derived functor of $\Gamma^{\mathrm{P}}(\mathrm{U},\text{-})$ is given by pointwise application of $\Gamma^{\mathrm{P}}(\mathrm{U},\text{-})$ since $\Gamma^{\mathrm{P}}(\mathrm{U},\text{-})$ is exact. Furthermore, ι preserves injective objects by corollary 1.33. Thus, the canonical morphism

$$R\,\Gamma(\mathrm{U},\text{-}) = R(\Gamma^{\mathrm{P}}(\mathrm{U},\text{-})\circ\iota) \to \Gamma^{\mathrm{P}}(\mathrm{U},\text{-})\circ R\iota$$

is an isomorphism. This proves the claim. □

Lemma 1.40. *Let $\{U_i \to U\}_{i\in I}$ be a covering in T and $F \in \mathrm{Sh}(T,\mathcal{C})$ be a sheaf such that*

$$\check{C}^p(U_{i_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{i_r}, \mathcal{F}) \cong 0$$

is zero for all $0 < p$, $0 \le r$ and $(i_0,...,i_r) \in I^{r+1}$. Then, the edge morphism

$$\check{C}^p(\{U_i \to U\},\mathcal{F}) \longrightarrow \mathrm{H}^p(U,\mathcal{F})$$

is an isomorphism for all p.

Proof. The cohomology groups

$$\mathrm{H}^p(\{U_i \to U\}, R^q\iota(\mathcal{F}))$$

are calculated by taking homology of the cochain complex $\check{C}^p(\{U_i \to U\}, R^q\iota(\mathcal{F}))$ by theorem 1.11. We obtain a natural sequence of isomorphisms

$$\begin{aligned}
\check{C}^p(\{U_i \to U\}, R^q\iota(\mathcal{F})) &= \prod_{(i_0,\dots,i_r)\in I^{r+1}} R^q\iota(\mathcal{F})(U_{i_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{i_r}) && \text{by definition}\\
&\cong \prod_{(i_0,\dots,i_r)\in I^{r+1}} \mathrm{H}^q(U_{i_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{i_r}, \mathcal{F}) && \text{by the above lemma}\\
&= 0 && \text{for } 0 < p.
\end{aligned}$$

Applied to the spectral sequence

$$\mathrm{E}_2^{pq} = \check{C}^p(\{U_i \to U\},-)(R^q\iota(\mathcal{F})) \Rightarrow \mathrm{E}^{p+q} = \mathrm{H}^{p+q}(U, \mathcal{F})$$

we conclude the claim. □

Lemma 1.41. *Assume there exists a converging* $\mathbb{N} \times \mathbb{N}$*-indexed spectral sequence*

$$\mathrm{E}_2^{pq} \Rightarrow \mathrm{E}^{p+q}$$

and an $n \in \mathbb{N}$ *such that* $\mathrm{E}_2^{pq} = 0$ *for all* $0 < q < n$. *Then, the edge morphism*

$$\mathrm{E}_2^{p0} \to \mathrm{E}^p$$

is an isomorphism for all $p < n$ *and a monomorphism for* $p = n$.

Proof. We observe $\mathrm{E}_r^{pq} = \mathrm{E}_2^{pq} = 0$ for all r and $0 < q < n$ since $\mathrm{E}_2^{pq} = 0$ for all $0 < q < n$. We deduce $\mathrm{E}_2^{p0} = \mathrm{E}_\infty^{p0}$ for all $p \leq n$ since the differentials d_r^{pq} have degree $(r, 1-r)$. We have $0 = \mathrm{E}_\infty^{p,m-p} \cong gr_p(\mathrm{E}^m)$ for all $m < n, p \neq m$ and for $m = n, p \neq 0$. Since the spectral sequence is converging, we deduce the desired. □

Corollary 1.42. *Let* $\mathcal{F} \in \mathrm{Sh}(T, \mathcal{C})$ *and* $U \in T$. *Assume*

$$\check{H}^p(U, R^q\iota(\mathcal{F})) \cong 0$$

is zero for all $0 < q < n$. *Then, the edge morphism*

$$\check{H}^p(U, \mathcal{F}) \longrightarrow \mathrm{H}^p(U, \mathcal{F})$$

is an isomorphism for all $p < n$ and a monomorphism for $p = n$.

Proof. We apply the previous lemma to the spectral sequence

$$\mathrm{E}_2^{pq} = \check{H}^p(U,-)(R^q\iota(\mathcal{F})) \Rightarrow \mathrm{E}^{p+q} = \mathrm{H}^{p+q}(U,\mathcal{F})$$

to obtain the claim. □

Proposition 1.43. *Let $\mathcal{F}$ be an object of* $\mathrm{Sh}(T,\mathcal{C})$*. Then,*

$$\check{H}^0(U, R^q\iota(\mathcal{F})) \cong 0$$

is zero for all $U \in T$, $0 < q$.

Proof. The induced morphism $\check{H}^0(U, R^q\iota(\mathcal{F})) \longrightarrow R^q\iota(\mathcal{F})^\#(U)$ is a monomorphism by theorem 1.17. Hence, it suffices to prove $R^q\iota(\mathcal{F})^\# \cong 0$ for all $U \in T$, $0 < q$. We apply Grothendieck's spectral sequence to the composition $(-)^\# \circ \iota \cong id_{\mathrm{Sh}(T,\mathcal{C})}$ to obtain a converging spectral sequence

$$\mathrm{E}_2^{pq} = R^p(-)^\#(R^q\iota(\mathcal{F})) \Rightarrow \mathrm{E}^{p+q} = R^{p+q}id_{\mathrm{Sh}(T,\mathcal{C})}(\mathcal{F}).$$

Then, the edge morphism

$$(R^q\iota(\mathcal{F}))^\# \longrightarrow R^q id_{\mathrm{Sh}(T,\mathcal{C})}(\mathcal{F})$$

is an isomorphism since $(-)^\#$ is exact. Furthermore, $(R^q\iota(\mathcal{F}))^\# \cong R^q id_{\mathrm{Sh}(T,\mathcal{C})}(\mathcal{F}) \cong 0$ is zero for all $0 < q$ since $id_{\mathrm{Sh}(T,\mathcal{C})}$ is exact. We deduce $R^q\iota(\mathcal{F})^\# \cong 0$ for all $U \in T$, $0 < q$ and, hence, the claim. □

Corollary 1.44. *Let $\mathcal{F} \in \mathrm{Sh}(T,\mathcal{C})$ be a sheaf. Then, the edge morphism*

$$\check{H}^p(U, \iota\mathcal{F}) \longrightarrow \mathrm{H}^p(U,\mathcal{F})$$

is an isomorphism for $p = 0, 1$ and a monomorphism for $p = 2$.

1.4.2 Flasque sheaves

Given some morphism of sites $f : T \to T'$ we would like to get a comparing spectral sequence of the sheaf cohomology groups of $\mathcal{F}$ and of $f_*\mathcal{F}$. We would like to apply Grothendieck's spectral sequence. To do so, we need to prove that f_* turns injective sheaves into $\Gamma(U,-)$-acyclic objects.

In this section, the distinction between a sheaf and its underlying presheaf becomes relevant. Therefore, we do not omit the inclusion $\iota : \mathrm{Sh}(T, \mathcal{C}) \subset \mathrm{PSh}(T, \mathcal{C})$ in this section.

Definition 1.45. A sheaf $\mathcal{F} \in \mathrm{Sh}(T, \mathcal{C})$ is called *flasque* if $\mathrm{H}^p(\{U_i \to U\}, \iota\mathcal{F}) = 0$ for all $0 < q$ and all coverings $\{U_i \to U\}_{i \in I}$.

Example 1.46. Let $I \in \mathrm{Sh}(T, \mathcal{C})$ be an injective sheaf. Then, $\iota I \in \mathrm{PSh}(T, \mathcal{C})$ is an injective sheaf. In particular,

$$\mathrm{H}^p(\{U_i \to U\}, \iota\mathcal{F}) \cong R^q \, \mathrm{H}^0(\{U_i \to U\}, \iota\mathcal{F}) \cong 0$$

is zero for all $q > 0$ since injective objects form an adapted class to every left exact functor and remark 1.36.

Let us recall a useful lemma.

Lemma 1.47. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume $\mathcal{A}$ has enough injective objects. Let $\mathcal{S}$ be a class of objects in $\mathcal{A}$ satisfying the following.*

1. *For all $A \in \mathcal{A}$ there exists a monomorphism $A \to C$ with $C \in \mathcal{S}$.*
2. *Given A, B in $\mathcal{A}$ such that its direct sum $A \oplus B$ is also in $\mathcal{A}$, then, both A and B are in $\mathcal{S}$.*
3. *Given some short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ and objects $A, B \in \mathcal{S}$, then, the induced sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact and C is in $\mathcal{S}$.

Then, $\mathcal{S}$ is adapted to F. In particular, resolutions of objects in $\mathcal{S}$ can be used to calculate RF. Furthermore, every injective object of $\mathcal{A}$ is in $\mathcal{S}$.

Proof. We only a sketch a proof. We check the conditions in remark 1.36. 1. and 3. in remark 1.36 hold trivial. Because $\mathcal{A}$ has enough injective objectets, RF exists. Let $K^\bullet$ be bounded below acyclic complex of objects in $\mathcal{S}$. Inductively we can prove that $K^\bullet$ splits into a family of short exact sequences of objects in $\mathcal{S}$ using that $\mathcal{S}$ is stable under quotients. Then, we can derive that F preserves exact bounded below complexes of objects in $\mathcal{S}$ using 3. This proves 2. of remark 1.36. At last, we can embed every injective sheaf into some object in $\mathcal{S}$ by 1. Thus, every injective sheaf is a split subobject of an object in $\mathcal{S}$. We derive every injective object to be in $\mathcal{S}$ by 2. □

Let us now check that the class of flasque sheaves satisfies all assumptions in lemma 1.47 for $F = \iota$.

Lemma 1.48. *Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence in $\mathrm{Sh}(T, \mathcal{C})$. *If* $\mathcal{F}'$ *is flasque, then,*

$$0 \longrightarrow \iota\mathcal{F}' \longrightarrow \iota\mathcal{F} \longrightarrow \iota\mathcal{F}'' \longrightarrow 0$$

is exact in $\mathrm{PSh}(T, \mathcal{C})$.

Proof. We need to prove

$$0 \longrightarrow \iota\mathcal{F}'(U) \longrightarrow \iota\mathcal{F}(U) \longrightarrow \iota\mathcal{F}''(U) \longrightarrow 0$$

to be exact for all $U \in T$. We deduce $\check{H}^1(U, \iota\mathcal{F}) \cong 0$ to be zero since

$$\check{H}^p(\{U_i \to U\}, \iota\mathcal{F}) \cong 0$$

is zero for all $0 < q$ and all coverings $\{U_i \to U\}_{i \in I}$. Then,

$$0 \cong \check{H}^1(U, \iota\mathcal{F}) \cong \mathrm{H}^1(U, \mathcal{F})$$

is zero by lemma 1.44. We deduce the claim by the long exact sequence in cohomology. □

Lemma 1.49. *Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence in $\mathrm{Sh}(T, \mathcal{C})$. *If* $\mathcal{F}'$ *and* $\mathcal{F}$ *are flasque, then,* $\mathcal{F}''$ *is flasque.*

Proof. By lemma 1.48,

$$0 \longrightarrow \iota\mathcal{F}' \longrightarrow \iota\mathcal{F} \longrightarrow \iota\mathcal{F}'' \longrightarrow 0$$

is exact in $\mathrm{PSh}(T, \mathcal{C})$. We obtain a long exact sequence of cohomology groups

$$0 \to \cdots \to \check{H}^p(\{U_i \to U\}, \iota\mathcal{F}') \to \check{H}^p(\{U_i \to U\}, \iota\mathcal{F}) \to \check{H}^p(\{U_i \to U\}, \iota\mathcal{F}'') \to \cdots$$

since $\check{H}^p(\{U_i \to U\}, \iota\mathcal{F}) \cong R^p\check{H}^0(\{U_i \to U\}, \iota\mathcal{F})$ are isomorphic. Using

$$\check{H}^p(\{U_i \to U\}\iota\mathcal{F}'') \text{ and } \check{H}^p(\{U_i \to U\}, \iota\mathcal{F}) = 0$$

are zero for all $p > 0$ we deduce the claim. □

Corollary 1.50. *The class of flasque sheaves is adapted to* ι*,* $\check{H}^0(\{U_i \to U\}, -)$ *and* $\Gamma(\mathrm{U}, \text{-})$*. In particular, flasque resolutions compute* $\check{H}^p(\{U_i \to U\}, -), R^p\iota$ *and* $\mathrm{H}^p(U, -)$*.*

Proof. It suffices to check the conditions in lemma 1.47. Since $\mathrm{Sh}(T, \mathcal{C})$ has enough injective sheaves and injective sheaves are flasque by example 1.46, we deduce 1. Furthermore,

$$\check{H}^p(\{U_i \to U\}, \iota(-)) = \mathrm{H}^p(\check{C}^\bullet(\{U_i \to U\}, \iota(-)))$$

commutes with finite direct sums since ι is left exact. In particular, 2. holds. Observe if 3. holds for ι, then, 3. holds for $\mathrm{H}^0(\{U_i \to U\}, -)$ and $\Gamma(\mathrm{U}, \text{-})$. At last, 3. for ι is essentially lemma 1.49 and lemma 1.48. □

1.4.3 Leray spectral sequence

In the following, let $f : T \longrightarrow T'$ and $g : T' \longrightarrow T''$ be morphisms of sites.

Lemma 1.51. *If* $\mathcal{F}' \in \mathrm{Sh}(T', \mathcal{C})$ *is flasque, then,* $f_*\mathcal{F}' \in \mathrm{Sh}(T, \mathcal{C})$ *is flasque.*

Proof. We need to prove $\check{H}^p(\{U_i \to U\}, f_*\mathcal{F}') \cong 0$ to be zero for all $0 < p$ and all coverings $\{U_i \to U\}_I$ in T. We obtain a natural isomorphism

$$\begin{aligned}\check{C}^p(\{U_i \to U\}, f_*\mathcal{F}') &= \prod_{(i_0,\dots,i_p)\in I^{p+1}} f_*\mathcal{F}'(U_{i_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{i_p}) \\ &\cong \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{F}'(f(U_{i_0}) \underset{f(U)}{\times} \cdots \underset{f(U)}{\times} f(U_{i_p})) \quad \text{by definition} \\ &= \check{C}^p(\{f(U_i) \to f(U)\}, \mathcal{F}').\end{aligned}$$

Hence,

$$\check{H}^p(\{U_i \to U\}, f_*\mathcal{F}') \cong \check{H}^p(\{f(U_i) \to f(U)\}, \mathcal{F}') \cong 0$$

is zero for all $0 < p$ since $\mathcal{F}'$ is flasque. □

Lemma 1.52. *The class of flasque sheaves is adapted to* $f_* : \mathrm{Sh}(T', \mathcal{C}) \to \mathrm{Sh}(T, \mathcal{C})$*.*

Proof. We check the conditions in lemma 1.47. Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence in $\mathrm{Sh}(T',\mathcal{C})$ with $\mathcal{F},\mathcal{F}'$ flasque. In order to prove f_* to preserve this exact sequence, it suffices to prove

$$0 \to \iota f_* \mathcal{F}' \to \iota f_* \mathcal{F} \to \iota f_* \mathcal{F}'' \to 0$$

to be exact in $\mathrm{PSh}(T,\mathcal{C})$. We easily deduce this from lemma 1.48. We already proved the rest of lemma 1.47 in the proof of corollary 1.50. □

Theorem 1.53 (Leray spectral sequence)**.** *Let $\mathcal{F}'' \in \mathrm{Sh}(T'',\mathcal{C})$ be a sheaf. There exists a converging spectral sequence*

$$\mathrm{E}_2^{pq} = R^p f_*(R^q g_*(\mathcal{F}'')) \Rightarrow R^{p+q}(f_* \circ g_*)(\mathcal{F}'') = \mathrm{E}^{p+q}$$

functorial in $\mathcal{F}''$.

Proof. We may apply Grothendieck's spectral sequence to $f_* \circ g_*$ by the previous lemma and by lemma 1.51. This is precisely the claim. □

Corollary 1.54. *Let $\mathcal{F}' \in \mathrm{Sh}(T',\mathcal{C})$ be a sheaf and U and object in T. There exists a converging spectral sequence*

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(U, R^q g_*(\mathcal{F}')) \Rightarrow \mathrm{H}^{p+q}(g(U),\mathcal{F}') = \mathrm{E}^{p+q}$$

functorial in $\mathcal{F}'$.

Proof. This is theorem 1.53 applied to $f : * \to T', pt \mapsto U$ of example 1.31. □

1.4.4 Compatibility with filtered colimits

A strong technic for deriving results in locally finitely presentable categories is to prove results for finitely presentable objects and then use every object to be a filtered colimit of such objects. To do so, we need to prove the result in question to be compatible with filtered colimits in a suitable sense.

Lemma 1.55. *Assume T is noetherian. Let $\{U_i \to U\}_{j\in J}$ be a covering in T. The functor $\check{H}^q(\{U_i \to U\}_{j\in J}, -)$ commutes with filtered colimits for all q.*

Proof. We may assume J to be finite. Let $(\mathcal{F}_i \to \mathcal{F})_I$ be a filtered colimit cocone in $\mathrm{Sh}(T,\mathcal{C})$. Recall the inclusion

$$\iota : \mathrm{Sh}(T,\mathcal{C}) \subset \mathrm{PSh}(T,\mathcal{C})$$

to preserve filtered colimits since T is noetherian. We deduce the induced morphism

$$\operatorname{colim}_I \check{H}^q(\{U_i \to U\}_{j\in J}, \iota\mathcal{F}_i) \to \check{H}^q(\{U_i \to U\}_{j\in J}, \iota\mathcal{F})$$

to be an isomorphism since filtered colimits are exact in $\mathcal{C}$. □

Corollary 1.56. *Let T be noetherian and $(\mathcal{F}_i \to \mathcal{F})_I$ be a filtered colimit cocone in $\mathrm{Sh}(T,\mathcal{C})$ with each $\mathcal{F}_i$ flasque. Then, $\mathcal{F}$ is flasque.*

Theorem 1.57. *Assume T is noetherian. Let $\mathcal{F}_(-) : I \to \mathrm{Sh}(T,\mathcal{C})$ be a filtered diagram and $(\mathcal{F}_i \to \mathcal{F})_I$ be a filtered colimit cocone in $\mathrm{Sh}(T,\mathcal{C})$. Then, the induced*

$$\operatorname{colim}_I \mathrm{H}^q(U,\mathcal{F}_i) \to \mathrm{H}^q(U,\mathcal{F})$$

is an isomorphism for all q. We say sheaf cohomology commutes with filtered colimits.

Proof. The category $\mathrm{Sh}(T,\mathcal{C})$ is locally finitely presentable (and abelian) by corollary 1.27. Thus, the category $\mathrm{Fun}(I,\mathrm{Sh}(T,\mathcal{C}))$ is locally finitely presentable and abelian. In particular, $\mathrm{Fun}(I,\mathrm{Sh}(T,\mathcal{C}))$ has enough injective objects by Lemma 4.11 in [6]. For every $i \in I$ define a functor

$$\tilde{i} : * \to I, pt \mapsto i.$$

We obtain a functor

$$\tilde{i}_* : \mathrm{Fun}(I,\mathrm{Sh}(T,\mathcal{C})) \to \mathrm{Fun}(*,\mathrm{Sh}(T,\mathcal{C})), \mathcal{G} \mapsto (pt \mapsto \mathcal{G}(i)).$$

Observe its left adjoint, given by the left Kan-extension of $\tilde{i}$, to be exact since I is filtered. In particular, $\tilde{i}_*$ preserves injective objects by lemma 1.32. We deduce evaluating at some i to preserve injective objects by identifying

$$\mathrm{Fun}(*,\mathrm{Sh}(T,\mathcal{C})) \simeq \mathrm{Sh}(T,\mathcal{C}), F \mapsto F(pt).$$

Let

$$0 \to \mathcal{F}_{(-)} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

be an injective resolution in $\mathrm{Fun}(I,\mathrm{Sh}(T,\mathcal{C}))$. Then, the complex

$$0 \to \mathcal{F}_i \to \mathcal{I}_0(i) \to \mathcal{I}_1(i) \to \cdots$$

is an injective resolution in $\mathrm{Sh}(T,\mathcal{C})$ for every i and computes sheaf cohomology of $\mathcal{F}_i$. We deduce the complex

$$0 \to \operatorname{colim}_I \mathcal{F}_i \to \operatorname{colim}_I \mathcal{I}_0(i) \to \operatorname{colim}_I \mathcal{I}_1(i) \to \cdots$$

to be a flasque resolution of $\mathcal{F}$ by corollary 1.56 and, therefore, computes sheaf cohomology by corollary 1.50. At last, each colimit is computed pointwise by lemma 1.26. Combined, the diagrams induced by the colimit cocone morphisms $\mathcal{I}_j(i) \to \operatorname{colim}_I \mathcal{I}_j(i)$ induce isomorphisms

$$\operatorname{colim}_I \mathrm{H}^q(U, \mathcal{F}_i) \to \mathrm{H}^q(U, \mathcal{F})$$

for every q. □

2 Étale Cohomology

2.1 Étale morphisms

We will very briefly introduce étale morphisms. Those will be an algebraic analogue of a local isomorphism in complex analytic topology. Let us give the probably most intuitive definition of an étale morphism of schemes.

Definition 2.1. A morphism of schemes $f : X \to S$ is *étale* if it is locally of finite presentation and for every $x \in X$ there exist $\mathrm{Spec}(A) = V \subset S$ and $x \in \mathrm{Spec}(B) = U \subset X$ affine opens such that $f(U) \subset V$ and the induced morphism of rings $A \to B$ exhibits

$$B \cong A[t_1, ..., t_n]/(f_1, ..., f_n)$$

to be isomorphic as A-algebras for some $n \in \mathbb{N}$ with $det((\partial f_i/\partial t_j)_{i,j}) \in B$ invertible in B_x. We say f is *étale at* $x \in X$ if there exist opens $V \subset S$ and $x \in U \subset X$ such that $f(U) \subset V$ and the induced $f : U \to V$ is étale.

A morphism of rings $A \to B$ is *standard étale* if

$$B \cong A[t]_g/(h)$$

are isomorphic as A-algebras with h being monic and its derivative $\partial h/\partial t = h'$ being invertible in B.

There are many equivalent characterizations of étale morphisms which are more familiar to work with. See for instance [10, Tag 02GU]. We only collect the results needed, as there are many well-prepared sources on this topic. See for instance [5] or [10, Tag 02GH].

Proposition 2.2. *Here is a collection of practical results on the étale morphisms of schemes.*

1. *The composition of étale morphisms of schemes is étale.*
2. *The base change of an étale morphism of schemes remains étale.*
3. *Étale morphisms are locally quasi-finite*
4. *Let X be a k-scheme for k a field. Then, X is an étale k-scheme iff X is a finite disjoint union of spectra of finite separable field extensions of k.*
5. *An étale morphism is flat.*
6. *An étale morphism of schemes is open.*

7. *Let $f : X \to S$ be a morphism of schemes and $x \in X$. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. Then, f is étale at x iff there exists an affine open $x \in U \subset X$ and $f(U) \subset V$ such that the induced morphism $f : U \to V$ is standard étale.*

8. *Let $f : X \to Y$ be a morphism of étale S-schemes. Then, f is étale.*

9. *A standard étale map of rings is étale.*

Proof.

1. [10, Tag 02GN]
2. [10, Tag 02GO]
3. [10, Tag 02WS]
4. [10, Tag 02GL]
5. [10, Tag 02GS]
6. [10, Tag 03WT]
7. [10, Tag 02GT]
8. [10, Tag 02GW]
9. Implicitely in (9) of [10, Tag 02GU]

□

Example 2.3. Let X be a scheme.

1. Every open immersion $U \subset X$ is étale by [10, Tag 02GP].
2. Let $X = \mathrm{Spec}(A)$ be an affine scheme for some ring A and $f \in A[t]$ be monic with f' and f jointly generating $A[t]$. Then, the canonical $A \to A[t]/(f)$ is standard étale, hence, étale.

2.2 The étale site and étale sheaves

Throughout the rest of this section let S be a scheme and $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}, {}_{\Lambda}\mathrm{Mod}$ for Λ some commutative ring.

Lemma 2.4 (Étale site)**.** *Define* $\mathrm{Et}_S \subset \mathrm{Sch}/S$ *to be the full subcategory of étale S-schemes. Then,* Et_S *has all fibre products, computed in* Sch/S*. Defining coverings in* Et_S *to be the families*

$$\{\phi_i : U_i \to U\}_{i \in I}$$

of S-morphisms with

$$\cup_{i \in I} \phi_i(U_i) = U$$

as sets defines a site which we also denote by Et_S*.*

Proof. By 1. and 2. of proposition 2.2 we deduce Et_S has all fibre products, computed in Sch/S. Recall the base change of a morphism of schemes surjective at the level of topological spaces remains surjective at the level of topological spaces. Combined, this proves 1. of definition 1.2. 2. and 3. of definition 1.2 hold trivially by using the composition of étale morphisms remains étale by 1. of proposition 2.2. □

Definition 2.5 (Étale and Zariski cohomology)**.** Let S be a scheme and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C})$ for $\mathcal{C} = \mathrm{Ab}, {}_{\Lambda}\mathrm{Mod}$. We define the *$p$-th étale cohomology group with values in $\mathcal{F}$* denoted by

$$\mathrm{H}^p_{et}(S, \mathcal{F}) = R^p\,\Gamma(\mathrm{S}, \text{-})(\mathcal{F})$$

to be the p-th right derived functor of global sections.

Let $\mathcal{G}$ be a sheaf on Zar_S. Denote by $\Gamma(\mathrm{S}, \text{-})_{\mathrm{Zar}}$ the global section functor on Zariski sheaves. We define

$$\mathrm{H}^p_{Zar}(S, \mathcal{G}) = R^p\,\Gamma(\mathrm{S}, \text{-})_{\mathrm{Zar}}(\mathcal{F})$$

to be the *p-th Zariski cohomology group with values in $\mathcal{G}$.* Observe this agrees with the classical Zariski cohomology group since sheaf cohomology on $\mathrm{Ouv}(X)$ and classical sheaf cohomology on a topological space X agree.

2.3 Examples of étale sheaves

We have a canonical inclusion $\mathrm{Zar}_X \subset \mathrm{Et}_X$ for every scheme X by example 2.3. Therefore, every étale sheaf is in particular a Zariski sheaf. However, being an étale sheaf is in general a stronger condition.

Lemma 2.6. *Let X be a scheme and $\mathcal{F} \in \mathrm{PSh}(\mathrm{Et}_X, \mathcal{C})$. Then, $\mathcal{F}$ is an étale sheaf iff it satisfies the sheaf condition for*

1. *coverings in* $\mathrm{Zar}_X \subset \mathrm{Et}_X$*.*
2. *coverings given by a single étale morphism of affine schemes surjective at the level of topological spaces.*

Proof. 3.1.1 Lemma in [11]. □

Let us recall a fundamental insight due to Grothendieck.

Definition 2.7. An fpqc (fidèlement plat et quasi-compact) morphism of schemes is a morphism of schemes $X \to S$ which is faithfully flat and quasi-compact.

Lemma 2.8. *Let $f : X \to X'$ be a fpqc morphism of S-schemes. Given another S-scheme Y, the induced diagram*

$$\mathrm{Hom}_{Sch/S}(X', S) \to \mathrm{Hom}_{Sch/S}(X', S) \rightrightarrows \mathrm{Hom}_{Sch/S}(X' \times_X X', S)$$

is an equalizer diagram. In other words, the canonical diagram

$$X' \times_X X' \rightrightarrows X' \to X$$

is a coequalizer diagram of S-schemes.

Proof. [10, Tag 03O3] □

Corollary 2.9. *Let X be an S-scheme. Then, the restriction of the presheaf $\mathrm{Hom}_S(-, X)$ to Et_S is an étale sheaf. In particular, every representable presheaf in $\mathrm{PSh}(\mathrm{Et}_X, \mathrm{Set})$ is a sheaf and the Yoneda embedding factorizes as*

$$Et_X \subset \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Set}) \subset \mathrm{PSh}(\mathrm{Et}_X, \mathrm{Set})$$

Proof. By the previous lemma, it suffice to proof the sheaf condition of $\mathrm{Hom}_S(-, X)$ only for 1. Zariski coverings and 2. coverings given by a single étale morphism of affine schemes surjective at the level of topological spaces. Part 1. is well known. Part 2. follows by lemma 2.8. □

Example 2.10. Let S be a scheme. We have seen the functor Γ(S,-) to admit a right adjoint in example 1.31. The left adjoint of Γ(S,-) maps a set resp. Λ-module M to the sheafification $(U \mapsto M)^{\#}$ of the constant presheaf induced by M by the very construction. We denote this sheaf by $\underline{M} : \mathrm{Et}_S \to \mathrm{Set}$ resp. $\underline{M} : \mathrm{Et}_S \to {}_{\Lambda}\mathrm{Mod}$.

Example 2.11. Let X be a scheme.

1. (additive group sheaf) The presheaf $U \mapsto \mathcal{O}_U(U)$ denoted by G_a is respresented by $\underline{\mathrm{Spec}}_X(\mathcal{O}_X[t])$ and, therefore, is an étale X-sheaf.
2. (multiplicative group sheaf) The presheaf $U \mapsto \mathcal{O}_U(U)^{\times}$ of abelian groups denoted by G_m is respresented by $\underline{\mathrm{Spec}}_X(\mathcal{O}_X[t, t^{-1}])$ and, therefore, is an étale X-sheaf.
3. (n-th roots of unity sheaf) The presheaf $U \mapsto \{u \in \mathcal{O}_U(U) | u^n = 1\}$ of abelian groups denoted by $\mu_{n,X}$ is respresented by $\underline{\mathrm{Spec}}_X(\mathcal{O}_X[t]/(t^n - 1))$ and, therefore, is an étale X-sheaf.

4. (constant sheaf) Given a set M, the associated constant étale X-sheaf $\underline{M} = (U \mapsto M)^{\#}$ is isomorphic to $U \mapsto \mathrm{Hom}_S(U, \sqcup_M X)$ for the (not necessarily étale) $\sqcup_M X \to X$.

2.4 Direct and inverse image functor

Remark 2.12. Let $f : X \to S$ be a morphism of schemes. Recall étale morphisms to be stable under base change. Thus,

$$\mathrm{Et}_S \to \mathrm{Et}_X, U \mapsto U \times_S X$$

is a functor. We convince ourselves that this functor preserves coverings. Therefore,

$$\mathrm{Et}_S \to \mathrm{Et}_X, U \mapsto U \times_S X$$

defines a morphism of sites.

Definition 2.13. Let $f : X \to S$ be a morphism of schemes. We call

$$f_* : \mathrm{Sh}(\mathrm{Et_X}, \mathcal{C}) \to \mathrm{Sh}(\mathrm{Et_S}, \mathcal{C}), \mathcal{F} \mapsto (U \mapsto \mathcal{F}(U \times_S X))$$

associated to $- \times_S X$ the *direct image of* f and its left adjoint

$$f^* : \mathrm{Sh}(\mathrm{Et_S}, \mathcal{C}) \to \mathrm{Sh}(\mathrm{Et_X}, \mathcal{C})$$

the *inverse image of* f.

Example 2.14. If $f : U \to S$ is étale, the associated inverse image f^* is given by restricting a sheaf on Et_S to Et_U along f. In this case, the counit of the adjunction $f^* \dashv f_*$ is an isomorphism.

Lemma 2.15. *Let $f : X \to S$ be a morphism of schemes and U an étale S-scheme. Then,*

$$f^*(h_U) \cong h_{U \times_Y X}$$

are canonically isomorphic. Furthermore, along this isomorphism, the unit $1 \Rightarrow f_* f^*$ exhibits pointwise as

$$- \times_S X : \mathrm{Hom}_S(V, U) \to \mathrm{Hom}_X(V \times_S X, U \times_S X).$$

Proof. We obtain natrual isomorphisms

$$\mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})}(f^*(h_U), -) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et_S}, \mathrm{Set})}(h_U, f_*(-)) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})}(h_{U \times_S X}, -)$$

by adjunction of $f^* \dashv f_*$. By Yoneda's Lemma, $f^*(h_U) \cong h_{U\times_Y X}$ are isomorphic. By the formula of the unit in remark 1.30, we deduce the second claim. □

Corollary 2.16. *Let $f : X \to S$ be a morphism of schemes and M be a finite set (resp. finite abelian group). Then,*

$$f^*\underline{M} \cong \underline{M}$$

are canonically isomorphic.

Proof. Recall the constant sheaf $\underline{M} \in \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$ (resp. $\underline{M} \in \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Ab})$) to be represented by $\sqcup_M X \to X$ by example 2.11. If M is finite, $\sqcup_M X \to X$ is étale. We deduce the claim by the previous lemma. □

Similar to ordinary sheaf cohomology, we can define a comparison morphism of cohomology groups for given $f : X \to Y$ a morphism of schemes. We will see in example 6.13 that this construction is a special case of the base change map defined in the last chapter.

Construction 2.17. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an abelian sheaf on Et_Y. Choose an injective resolution $\mathcal{F} \to \mathrm{I}^\bullet$. Choose a quasi-isomorphism $f^*\,\mathrm{I}^\bullet \to \mathrm{J}^\bullet$ to an injective complex. Because f^* is exact, $f^*\,\mathcal{F} \to \mathrm{J}^\bullet$ is an injective resolution. Let ϵ be the unit of the adjunction $f^* \dashv f_*$. We obtain a morphism of complexes*

$$\mathrm{I}^\bullet(Y) \to f_*f^*\,\mathrm{I}^\bullet(Y) \cong f^*\,\mathrm{I}^\bullet(X) \to \mathrm{J}^\bullet(X)$$

and, therefore, a morphism of cohomology groups

$$\mathrm{H}^q_{et}(Y, \mathcal{F}) \to \mathrm{H}^q_{et}(X, f^*\,\mathcal{F}).$$

Combined with the functoriality of $\mathrm{H}^q_{et}(X, -)$, given a morphism of sheaves $\alpha : f^*\,\mathcal{F} \to \mathcal{G}$, we obtain a morphism

$$\mathrm{H}^q_{et}(Y, \mathcal{F}) \to \mathrm{H}^q_{et}(X, \mathcal{G}).$$

Observe this map to be given by the adjoint $\mathcal{F} \to f_*\,\mathcal{G}$ of α on $q = 0$.

Remark 2.18. Recall the first étale cohomology group and the first Čech cohomology group to be isomorphic by corollary 1.44. Along this isomorphism, the base change

$$\mathrm{H}^1_{et}(Y, -) \to \mathrm{H}^1_{et}(X, f^*(-))$$

induces a morphism

$$\check{H}^1(Y, -) \to \check{H}^1(X, f^*(-))$$

of functors on sheaves. We would expect this morphism to be induced by

$$\check{C}^\bullet(\{U_i \to Y\}, \mu_{\mathcal{F}}) : \check{C}^\bullet(\{U_i \to Y\}, \mathcal{F}) \to \check{C}^\bullet(\{U_i \to Y\}, f_* f^* \mathcal{F}) = \check{C}^\bullet(\{U_i \times_S X \to X\}, f^* \mathcal{F})$$

for $\mu : 1 \Rightarrow f_* f^*$ the unit of the adjunction $f^* \dashv f_*$. We will proof this to be true in corollary 6.14.

2.5 Categorical properties

Definition 2.19. Let X be a scheme. Define

$$\mathrm{Et}_X^{fp} \subset \mathrm{Et}_X$$

to be the full subcategory of finitely presented and étale morphisms. Being of finite presentation is stable under base change and composition. Therefore, Et_X^{fp} has fibre products computed in Et_X. We equip Et_X^{fp} with the unique structure of a site by declaring a family to be a covering iff it is a covering in Et_X. In particular, the inclusion $\mathrm{Et}_X^{fp} \subset \mathrm{Et}_X$ is a morphism of sites.

Lemma 2.20. *Assume S to be quasi-compact. Then, Et_S^{fp} is noetherian.*

Proof. Let

$$\{\phi_i : U_i \to U\}_{i \in I}$$

be a covering in Et_S^{fp}. We need to prove that there exists some $J \subset I$ finite such that

$$\{\phi_i : U_i \to U\}_{i \in J}$$

is a covering. Every étale $U_i \to U$ is in particular open. Since U is a quasi-compact S-scheme and S is quasi-compact, U is quasi-compact. Thus, there exists a finite subcovering

$$\{\phi_i(U_i) \subset U\}_{i \in I}$$

of the open topological covering

$$\{\phi_i(U_i) \subset U\}_{i \in I}.$$

Then,

$$\{\phi_i : U_i \to U\}_{i \in J}$$

is a finite subcovering. □

Lemma 2.21. *Assume X to be quasi-separated. Then, the inclusion* $\mathrm{Et}_X^{fp} \subset \mathrm{Et}_X$ *induces an equivalence*

$$\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \simeq \mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathcal{C}).$$

Proof. It suffices to prove that every $U \to X$ étale admits a covering by schemes étale and of finite presentation over X. Then, we can deduce every sheaf as well as every morphism of sheaves to be fully determined by its restriction to Et_X^{fp} by using the sheaf property. For every $u \in U$ there exists an affine open neighbourhood $u \in U_u \subset U$ and an affine open $V_u \subset X$ such that $U_u \to X$ factorizes through $V_u \subset X$ and $U_u \to V_u$ is of finite presentation. We claim the étale map $U_u \to X$ to be of finite presentation. It suffices to prove $V_u \subset X$ to be of finite presentation, i.e. to prove the inclusion to be quasi-compact since open immersios are locally of finite presentation and quasi-separated. Given some $V \subset X$ open and quasi-compact, we may cover V by finitely many affine opens and assume V to be affine. Because X is quasi-separated, the preimage of V in V_u, which is $V \cap V_u$, may be covered by finitely many affine opens and is, therefore, quasi-compact. We deduce the claim. □

Corollary 2.22. *Assume X to be quasi-compact and quasi-separated. Then,* $\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C})$ *is locally finitely presentable and filtered colimits are computed pointwise for every $U \in \mathrm{Et}_X^{fp}$. Furthermore, a strong generator of finitely presentable objects in* $\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$ *is given by* $\mathrm{Et}_X^{fp} \subset \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$.

Proof. Combining the previous two lemmata, we obtain that the restriction induces an equivalence

$$\mathrm{Sh}(\mathrm{Et_X}, \mathcal{C}) \simeq \mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathcal{C})$$

and Et_X^{fp} is a noetherian site. In particular, $\mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathcal{C})$ to be locally finitely presentable and a strong generator of finitely presentable objects is given by

$$\mathrm{Et}_X^{fp} \subset \mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathrm{Set})$$

by corollary 1.27. At last, filtered colimits in $\mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathrm{Set})$ are computed pointwise by lemma 1.26. □

Lemma 2.23. *The Yoneda embedding preserves finite coproducts.*

Proof. Let $U_1, ..., U_n$ be étale X-schemes. Then, the induced $\sqcup_{i=1}^n U_i \to X$ is étale and the induced family $\{U_j \to \sqcup_{i=1}^n U_i\}_{j=1,...,n}$ is an étale covering. We need to prove the induced morphism

$$\mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C})}(h_{\sqcup_{i=1}^n U_i}, \mathcal{F}) \to \prod_{i=1}^n \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C})}(h_{U_i}, \mathcal{F})$$

to be an isomorphism for every $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Set})$. By Yoneda's Lemma, this is equivalent to proving the induced

$$\mathcal{F}(\sqcup_{i=1}^n U_i) \to \prod_{i=1}^n \mathcal{F}(U_i)$$

to be an isomorphism. This is the sheaf condition of $\mathcal{F}$ for the covering

$$\{U_j \to \sqcup_{i=1}^n U_i\}_{j=1,\dots,n}.$$

We deduce the claim. □

Proposition 2.24. *Let $U \in \mathrm{Et}_X$ and $\mathcal{F} \subset h_U$ be a subsheaf in* $\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$*. Then, there exists some open subscheme $V \subset U$ and an isomorphism $h_V \cong \mathcal{F}$.*

Proof. (ii) of Proposition 5.2.7 in [2]. □

Corollary 2.25. *Let X be quasi-compact and quasi-separated. Then, $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Set})$ is finitely presentable iff there exists some $U \in \mathrm{Et}_X^{fp}$ and an epimorphism $h_U \to \mathcal{F}$.*

Proof. By corollary 2.22, $\mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathrm{Set})$ is locally finitely presentable with strong generator

$$\mathrm{Et}_X^{fp} \subset \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Set}).$$

Thus, every finitely presentable object in $\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$ is a finite colimit of objects in Et_X^{fp}. We deduce the finitely presentable objects in $\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$ to be the coequalizers of sheaves representable by objects in Et_X^{fp} since $\mathrm{Et}_X^{fp} \subset \mathrm{Sh}(\mathrm{Et}_X^{fp}, \mathrm{Set})$ is closed under finite coproducts. In particular, every finitely presentable object is a quotient of a representable sheaf. On the other hand, given some epimorphism $h_U \to \mathcal{F}$, we obtain an induced coequalizer diagram

$$h_U \times_{\mathcal{F}} h_U \rightrightarrows h_U \to \mathcal{F}\,.$$

The canonical

$$h_U \times_{\mathcal{F}} h_U \to h_U \times h_U \cong h_{U \times_X U}$$

is a monomorphism. Thus, we deduce $h_U \times_{\mathcal{F}} h_U$ to be representable by lemma 5.5. Thus, $\mathcal{F}$ is a coequalizer of representable sheaves. We deduce the claim. □

Using the theory of relative normalization, we can proof the following.

Theorem 2.26. *Assume X is a noetherian scheme and Λ is a noetherian ring. Let $\mathcal{F}$ be a finitely presentable object in* $\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$ *resp.* $\mathrm{Sh}(\mathrm{Et_X}, {}_\Lambda\mathrm{Mod})$*. Then, there exist*

$n \in \mathbb{N}$*, finite and over a non-empty open étale morphisms* $f_i : Y_i \to X$*, finite sets resp. finite* Λ*-modules* E_i *for all* $i = 1, ..., n$ *and a monomorphism*

$$\mathcal{F} \subset \prod_{i=1}^{n} f_{i*}\underline{E_i}.$$

Proof. This is [10, Tag 09Z6] where finitely presentable is replaced by constructible. However, we deduce the equivalence of both conditions, being constructible and being finitely presentable, for sheaves of sets by [10, Tag 09Y9] combined with the arguments in corollary 2.25.

The constructible sheaves of Λ-modules are precisely the finite colimits of objects of the form $\Lambda[-] \circ h_U$ for $U \in \mathrm{Et}_X^{fp}$ by [10, Tag 095N]. Therefore, it suffices to prove that such objects define a strong generator of finitely presentable objects in $\mathrm{Sh}(\mathrm{Et}_X, {}_\Lambda\mathrm{Mod})$. The forgetful functor

$${}_\Lambda\mathrm{Mod} \to \mathrm{Set}$$

preserves filtered colimits. We deduce the forgetful functor

$$\mathrm{Sh}(\mathrm{Et_X}, {}_\Lambda\mathrm{Mod}) \to \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$$

to preserve filtered colimits by using corollary 2.22. Thus, given $U \in \mathrm{Et}_X^{fp}$, the Λ-module sheaf $\Lambda[-] \circ h_U$ is finitely presentable since h_U is by using

$$\mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et_X}, {}_\Lambda\mathrm{Mod})}(\Lambda[-] \circ h_U, -) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})}(h_U, -).$$

To check that such sheaves form a strong generator is straight forward by using that representable sheaves form a strong generator. □

The proof given in [10, Tag 09Z6] rests on the ascending chain condition for finitely presentable objects. Let us as at least give a proof for representable sheaves of sets.

Lemma 2.27. *Let* X *be a noetherian scheme and* $U \to X$ *étale and quasi-compact. Then,* $h_U \in \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Set})$ *satsifies the ascending chain condition, i.e. for every ascending chain of subsheaves*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset h_U$$

there exists some i_0 *with* $\mathcal{F}_i = \mathcal{F}_{i+1}$ *for all* $i \geq i_0$*.*

Proof. Let

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset h_U$$

be an ascending chain of subsheaves. Then, there exist $U_i \subset U$ open such that $\mathcal{F}_i \cong h_{U_i}$ by proposition 2.24. Remark U to be noetherian since it is locally of finite presentation and quasi-compact over a noetherian scheme. Thus, the open subset $\cup_{\mathbb{N}} U_i \subset U$ is quasi-compact. In particular, the open covering

$$\{U_i \subset \cup_{\mathbb{N}} U_i\}_{\mathbb{N}}$$

admits a finite subcovering

$$\cup_{\mathbb{N}} U_i = U_{i_1} \cup \cdots \cup U_{i_n}.$$

We conclude $U_i = U_{i+1}$ for every $i \geq max\{i_1, ..., i_n\}$. □

2.6 Limits of schemes

Here is a pleasant property of finitely presentable algebras. Let

$$(f_i : A_i \to A)_I$$

be a filtered colimit cocone of rings. By the very construction of the filtered colimit, for every finite collection of elements

$$a_1, ..., a_n \in A$$

there exists some $i \in I$ and

$$a_{i1}, ..., a_{in} \in A_i$$

such that $f_i(a_{ik}) = a_k$. Consider the finitely presentable A-algebra

$$B \cong A[t_1, ..., t_n]/(g_1, ..., g_m).$$

There exists some i such that the finitely many coefficients of all g_j are in the image of f_i. We deduce the existence of some A_i-algebra B_i of finite presentation together with an isomorphism

$$A \otimes_{A_i} B_i \cong B.$$

Using this idea, we can even prove every morphism of finitely presentable A-algebras to be induced by some morphism of finitely presentable A_i-algebras in the above sense. We can even generalize the above to cofiltered limits of schemes.

Definition 2.28. Let S be a scheme. Define

$$\mathrm{Sch}/S^{fp} \subset \mathrm{Sch}/S$$

to be the full subcategory of S-schemes of finite presentation.

The category of schemes does not have all limits. However, by using the relative spectrum construction, we may prove the existence of limits of cofiltered diagrams with affine transition maps, see [10, Tag 01YX] for the directed set case.

Theorem 2.29. *Let* $S_{(-)} : I \to \operatorname{Sch}/S_0$ *be a cofiltered diagram with affine transition maps and* S_0 *quasi-compact and quasi-separated. Define* $(\phi_i : S \to S_i)_I$ *to be its limit in* Sch/S_0*. Then, the induced functor*

$$\operatorname*{colim}_I(\operatorname{Sch}/{S_i}^{fp}) \xrightarrow{\sim} \operatorname{Sch}/S^{fp}, [(X_i, S_i)] \mapsto S \times_{S_i} X_i$$

is an equivalence.

Proof. We see by 1.5 Theorem in [7] that there always exists a small cofinal subdiagram of a filtered diagram given by a directed set. Therefore, we may assume I is a directed set. Then, we deduce the existence of the limit of $S_{(-)}$ by [10, Tag 01YX] and the second claim by [10, Tag 01ZM]. □

Proposition 2.30. *By the very nature of the proof, the equivalence restricts to an equivalence of full subcategories with property (P), for (P) being*

1. *open immersion*
2. *closed immersion*
3. *separated*
4. *finite*
5. *surjective at the level of topological spaces*
6. *étale*
7. *proper*

Proof. This is Proposition 1.10.10 in [2] despite 6. For 6. see [10, Tag 07RP]. □

Theorem 2.31. *Let* $S_{(-)} : I \to \operatorname{Sch}/S_0$ *be a cofiltered diagram of quasi-compact and quasi-separated* S_0*-schemes with affine transition maps and* S_0 *quasi-compact and quasi-separated. Denote by*

$$(\phi_i : S \to S_i)_I$$

its limit in Sch/S_0*. Assume all* S*,* S_0 *and* S_i *to be noetherian. Let*

$$\mathcal{F} \in \operatorname{Sh}(\mathrm{Et}_{\mathrm{S}}, {}_{\Lambda}\operatorname{Mod})$$

be finitely presentable for Λ *some noetherian ring. Then, there exists some*

$$\mathcal{F}_i \in \mathrm{Sh}(\mathrm{Et}^{fp}_{S_i}, {}_\Lambda\mathrm{Mod})$$

finitely presentable and an isomorphism

$$\phi_i^* \mathcal{F}_i \cong \mathcal{F}\,.$$

Proof. This is Lemma 5.9.8 in [2] where finitely presentable is replaced by constructible. However, we argue similar as in the proof of theorem 2.26 to deduce the claim. □

2.7 Étale Cohomology generalizes cohomology of quasi-coherent sheaves

In this chapter we briefly state the comparison of étale and quasi-coherent sheaf cohomology. We will often omit the distinction between a quasi-coherent module and its underlying sheaf of abelian groups.

The étale sheaf G_a factorizes canonically through the category of commutative rings. To indicate this, we denote it by

$$\mathcal{O}_{S,et} : \mathrm{Et}_X \to \mathrm{CRing}, U \mapsto \mathcal{O}_U(U).$$

Observe the restriction of $\mathcal{O}_{S,et}$ to Zar_S and the structure sheaf $\mathcal{O}_S$ to agree.

Lemma 2.32. *Let* S *be a scheme and* $\mathcal{M}$ *be a quasi-coherent* $\mathcal{O}_S$*-module. Denote by* g_q^* *the pullback of quasi-coherent modules, i.e.* $g_q^*\mathcal{M} = \mathcal{O}_X \otimes_{g^*\mathcal{O}_S} g^*\mathcal{M}$ *for every morphism of schemes* $g : X \to S$*. Then, the presheaf*

$$\mathcal{M}_{et} : \mathrm{Et}_S \to \mathrm{Ab}, (f : U \to S) \mapsto \Gamma(U, f_q^*\mathcal{M})$$

is a sheaf.

Proof. Example 1.2.6.1 in [12]. □

Example 2.33. By the very construction, $\mathcal{O}_{S,et} \cong (\mathcal{O}_S)_{et}$ are canonically isomorphic.

Corollary 2.34. *Let* S *be a scheme and* $\mathcal{M}$ *be a quasi-coherent* $\mathcal{O}_S$*-module. Then, the restriction of* $\mathcal{M}_{et}$ *to* $\mathrm{Zar}_S \subset \mathrm{Et}_S$ *is canonically isomorphic to* $\mathcal{M}$*.*

Remark 2.35. Given a scheme S, denote by

$$i : \mathrm{Zar}_S \to \mathrm{Et}_S$$

the inclusion of sites. Observe that the restriction

$$i_* : \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Ab}) \to \mathrm{Sh}(\mathrm{Zar}_S, \mathrm{Ab})$$

preserves flasque sheaves and is left exact. Then, Grothendieck's spectral sequence applied to the composition $\Gamma(\mathrm{S},\text{-}) = \Gamma(\mathrm{S},\text{-})_{Zar} \circ i_*$ of left exact functors yields a converging spectral sequence

$$E_2^{pq} = \mathrm{H}^p_{Zar}(S, R^q i_* \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}_{et}(S, \mathcal{F}) = E^{p+q}$$

for every $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Ab})$. Identifying

$$R^0 i_* \mathcal{M}_{et} = i_* \mathcal{M}_{et} \cong \mathcal{M}$$

by corollary 2.34 the spectral sequence yields an edge morphism

$$\mathrm{H}^p_{Zar}(S, \mathcal{M}) \to \mathrm{H}^p_{et}(S, \mathcal{M}_{et}).$$

Recall the following central lemma.

Lemma 2.36. *Let $X = \mathrm{Spec}(A)$ be an affine scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then, $\mathrm{H}^q_{\mathrm{Zar}}(X, \mathcal{M}) = 0$ for all $q > 0$.*

Proof. Theorem 18.2.4 in [4]. □

Theorem 2.37. *Let S be a scheme and $\mathcal{M}$ be a quasi coherent $\mathcal{O}_S$-module. Then, the edge morphism in remark 2.35*

$$\mathrm{H}^p_{Zar}(S, \mathcal{M}) \to \mathrm{H}^p_{et}(S, \mathcal{M}_{et})$$

is an isomorphism for all p.

Proof. In order to prove

$$\mathrm{H}^p_{Zar}(S, \mathcal{M}) \to \mathrm{H}^p_{et}(S, \mathcal{M}_{et})$$

to be an isomorphism it suffices to prove $R^q i_* \mathcal{M}_{et} = 0$ for all $q > 0$ by using the converging spectral sequence from above. Recall $R^q i_* \mathcal{M}_{et}$ to be given by the sheafification of the Zariski presheaf

$$\mathrm{Zar}_S \to \mathrm{Ab}, (U \subset S) \mapsto \mathrm{H}^q_{Zar}(U, i_* \mathcal{M}_{et})$$

by lemma 1.34. It suffices to prove

$$\mathrm{H}^q_{Zar}(U, \mathcal{M}) \cong \mathrm{H}^q_{Zar}(U, i_* \mathcal{M}_{et}) \cong 0$$

for every affine open $U \subset S$ and $q > 0$ since every open $U \subset S$ admits a covering by affine opens. We deduce the claim by lemma 2.36. □

Using similar arguments we can prove the following lemma.

Lemma 2.38. *Let X be a scheme. Then, the edge morphism*

$$\mathrm{H}^1_{Zar}(X, \mathcal{O}_X^\times) \to \mathrm{H}^1_{et}(X, \mathcal{O}_{X,et}^\times)$$

is an isomorphism.

Proof. Proposition 5.7.7. in [2]. □

Remark 2.39. Let $f : X \to S$ be a morphism of schemes. We obtain a canonical morphism

$$\mathcal{O}_{S,et}^\times \to f_* \mathcal{O}_{X,et}^\times$$

such that the restriction to Zar_S is the canonical

$$\mathcal{O}_S^\times \to f_* \mathcal{O}_X^\times .$$

Construction 2.17 yields a morphism

$$\mathrm{H}^1_{et}(S, \mathcal{O}_{S,et}^\times) \to \mathrm{H}^1_{et}(X, \mathcal{O}_{X,et}^\times).$$

Along the isomorphism of lemma 2.38, we obtain a morphism

$$\alpha : \mathrm{H}^1_{Zar}(S, \mathcal{O}_S^\times) \to \mathrm{H}^1_{Zar}(X, \mathcal{O}_X^\times).$$

We can check that this is the base change in topological sheaf cohomology. Recall furthermore $\mathrm{Pic}(-) \cong \mathrm{H}^1_{\mathrm{Zar}}(-, \mathcal{O}_{(-)}^\times)$ to be isomorphic such that the diagram

$$\begin{tikzcd}
\mathrm{Pic}(S) \arrow[r, phantom, "\cong"] \arrow[d, "f^*(-)\otimes_{f^* \mathcal{O}_S} \mathcal{O}_X"'] & \mathrm{H}^1_{\mathrm{Zar}}(S, \mathcal{O}_S^\times) \arrow[d, "\alpha"] \\
\mathrm{Pic}(X) \arrow[r, phantom, "\cong"] & \mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{O}_X^\times)
\end{tikzcd}$$

commutes where the left vertical arrow is given by the base change of quasi-coherent modules.

Lemma 2.40. *Let X be a scheme and I be a coherent $\mathcal{O}_X$-ideal with*

$$I^n = 0$$

for some n. Denote by

$$F : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}_X\,/I) \to X$$

the induced morphism. If $H^2_{Zar}(X, I) = 0$, then, base changing induces an epimorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$. *If $H^1_{Zar}(X, I) = 0$, then, base changing induces a monomorphism* $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$.

Proof. Let $n \in \mathbb{N}$ such that $I^n = 0$. The canonical morphism $X' \to X$ factorizes as a canonical sequence

$$\underline{\mathrm{Spec}}_X(\mathcal{O}_X\,/I) \to \underline{\mathrm{Spec}}_X(\mathcal{O}_X\,/I^2) \to \cdots \to \underline{\mathrm{Spec}}_X(\mathcal{O}_X\,/I^{n-1}) \to X.$$

By induction, we may assume $n = 2$. Observe that the topological spaces of X and X' agree since the defining ideal is nilpotent. We obtain a short exact sequence of abelian sheaves on the topological space X

$$0 \to I \to \mathcal{O}_X^\times \to \mathcal{O}_{X'}^\times \to 0$$

where the first morphism is given by $\alpha \mapsto 1 + \alpha$ and the second morphism is the canonical $\mathcal{O}_X^\times \to f_*\,\mathcal{O}_{X'}^\times = \mathcal{O}_{X'}^\times$. The claim follows by the long exact sequence of Zariski cohomology groups and lemma 2.38 together with the compatibility of remark 2.39. □

Corollary 2.41. *If $X = \mathrm{Spec}(A)$ is the spectrum of a commutative ring, then,*

$$H^2_{Zar}(X, I) = H^1_{Zar}(X, I) = 0$$

by lemma 2.36. Hence, wo obtain the well known isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(X')$.

Corollary 2.42. *Let X be a scheme and I be a coherent $\mathcal{O}_X$-ideal sheaf with $I^n = 0$ for some n. Denote by*

$$X' = \underline{\mathrm{Spec}}_X(\mathcal{O}_X\,/I).$$

If X is a noetherian scheme of dimension ≤ 1, then,

$$H^2_{Zar}(X, I) = 0$$

and, hence, $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ *is an epimorphism.*

Proof. By Theorem III.2.7 in [13] we deduce $\mathrm{H}^q_{\mathrm{Zar}}(X, I) = 0$ for all $q > dimX = 1$. We deduce the claim by lemma 2.40. □

2.8 Compatibility with filtered colimits

Theorem 2.43. *Let S be a quasi-compact and quasi-separated scheme and $(\mathcal{F}_i \to \mathcal{F})_I$ be a colimit cocone in $\mathrm{Sh}(\mathrm{Et}_\mathrm{S}, \mathcal{C})$ for some filtered index category I and $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}$. The induced cocone*

$$(\mathcal{F}_i(U) \to \mathcal{F}(U))_I$$

is a colimit cocone for all $U \in \mathrm{Et}^{fp}_S$. If $\mathcal{C} = \mathrm{Ab}$, then, the induced cocone

$$(\mathrm{H}^q_{et}(U, \mathcal{F}_i) \to \mathrm{H}^q_{et}(U, \mathcal{F}))_I$$

is a colimit cocone for all $U \in \mathrm{Et}^{fp}_S$ and q.

Proof. The restriction $\mathrm{Sh}(\mathrm{Et}_\mathrm{S}, \mathrm{Ab}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{Et}^{fp}_S, \mathrm{Ab})$ defines an equivalence by lemma 2.20. Furthermore, Et^{fp}_S is noetherian by lemma 2.21. We deduce the claim by theorem 1.57. □

We would in addition like to pass cofiltered limits of schemes through cohomology groups in order to get noetherian arguments into the game. Let us provide a framework for that.

Fix a cofiltered diagram

$$S_{(-)} : I^{op} \to \mathrm{Sch}\,/S_0$$

of quasi-compact and quasi-separated S_0-schemes with affine transition maps and S_0 quasi-compact and quasi-separated. Denote by

$$(\phi_i : S \to S_i)_{I^{op}}$$

its limit in $\mathrm{Sch}\,/S_0$. The main observation is that a sheaf on S is completely determined by its direct images $\phi_{i*}\mathcal{F}$ by theorem 2.29. As a convention we will not distinguish $\phi^P_{i,*} : \mathrm{PSh}(\mathrm{Et}_S, \mathcal{C}) \to \mathrm{PSh}(\mathrm{Et}_{S_i}, \mathcal{C})$ and its restrictions to sheaves $\phi_{i,*}$ whenever it is not necessary.

Definition 2.44. We define the *category of presheaves on $S_{(-)}$*, denoted by $\mathrm{PSh}(S_{(-)}, \mathrm{Set})$ as follows.

1. Objects are families $(\mathcal{F}_i)_{i \in I}$ of presheaves $\mathcal{F}_i \in \mathrm{PSh}(\mathrm{Et}_{S_i}, \mathrm{Set})$ together with mor-

phisms $\alpha_f : \mathcal{F}_j \to S_{f*}\mathcal{F}_i$ for every $f : i \to j$ in I^{op} such that

$$\begin{tikzcd}
\mathcal{F}_k \arrow[r, "\alpha_g"] \arrow[d, "\alpha_{f\circ g}"] & (S_g)_*\mathcal{F}_j \arrow[d, "(S_g)_*(\alpha_f)"] \\
(S_{f\circ g})_*\mathcal{F}_i \arrow[r, "\cong"] & (S_f)_* \circ (S_g)_*\mathcal{F}_i
\end{tikzcd}$$

commutes for every $f : i \to j, g : j \to k$ in I^{op}. We denote such an object by $(\mathcal{F}_i, \alpha_f)$.

2. Morphisms $(\mathcal{F}_i, \alpha_f) \to (\mathcal{G}_i, \beta_f)$ are families of morphisms $(f_i : \mathcal{F}_i \to \mathcal{G}_i)_{i\in I}$ of presheaves such that

$$\begin{tikzcd}
\mathcal{F}_j \arrow[r, "f_j"] \arrow[d, "\alpha_f"] & \mathcal{G}_j \arrow[d, "\beta_f"] \\
S_{f*}\mathcal{F}_i \arrow[r, "S_{f*}(f_i)"] & S_{f*}\mathcal{G}_i
\end{tikzcd}$$

commutes.

3. The composition is given pointwise and the identity is the canonical family of identities.

One can check that this defines a category.

Remark 2.45. We obtain a canonical functor

$$R : \mathrm{PSh}(\mathrm{Et}_S, \mathrm{Set}) \to \mathrm{PSh}(S_{(-)}, \mathrm{Set}), \mathcal{F} \mapsto (\phi_{i*}\mathcal{F}, \alpha_f)$$

with

$$\alpha_f : (\phi_j)_*\mathcal{F} \to (S_f \circ \phi_i)_*\mathcal{F}$$

induced by the canonical isomorphism $\phi_j \cong S_f \circ \phi_i$ for every $f : j \to i$ in I. Let

$$(F_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$$

be a presheaf on $S_{(-)}$. For each $U \in \mathrm{Et}_S^{fp}$ there exists some i and $U_i \in Et_{S_i}^{fp}$ such that $U \cong S \times_{S_i} U_i$ by proposition 2.30. Define

$$U_j = U_i \times_{S_i} S_j$$

for every $j \in i/I$. By fully faithfulness in proposition 2.30 and filteredness of I it is straight forward to check that $\mathrm{colim}_{i/I}\mathcal{F}_j(U_j)$ is (up to natural isomorphism) independent of the choice of i and U_i. Thus, the assignement $U \mapsto \mathrm{colim}_{i/I}\mathcal{F}_i(U_i)$ is well defined. Given some morphism $U \to V$ in Et_S^{fp} there exists some j and $U_j \to V_j$ in $\mathrm{Et}_{S_j}^{fp}$ inducing $U \to V$ by fully faithfulness in theorem 2.29. We obtain a unique morphism

$$\mathrm{colim}_{I/j}\, \mathcal{F}_k(V_k) \to \mathrm{colim}_{I/j}\, \mathcal{F}_k(U_k).$$

Thus, the assignement extends to a presheaf denoted by $L((\mathcal{F}_i, \alpha_f))$ by using lemma 2.21. We can check this to extend canonically to a functor

$$L : \mathrm{PSh}(S_{(-)}, \mathrm{Set}) \to \mathrm{PSh}(\mathrm{Et}_S, \mathrm{Set}).$$

Remark 2.46. Observe

$$\phi_{i*} L((\mathcal{F}_i, \alpha_f))(U) \cong \mathrm{colim}_{f \in i/I}((S_f)_* \mathcal{F}_j(U))$$

to be canonically isomorphic for all $U \in \mathrm{Et}_{S_i}^{fp}$ by definition. We deduce

$$\phi_{i*} L((\mathcal{F}_i, \alpha_f)) \cong \mathrm{colim}_{f \in i/I}(S_f)_* \mathcal{F}_j$$

to be canonically isomorphic. Therefore, if all $\mathcal{F}_i$ are sheaves, then, also $\phi_{i*} L((\mathcal{F}_i, \alpha_f))$ is a sheaf by theorem 2.43.

Lemma 2.47. *The functor R is right adjoint to L constructed in remark 2.45.*

Proof. We deduce a natural bijection

$$\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_S, \mathrm{Set})}(L((\mathcal{F}_i, \alpha_f)), \mathcal{G}) \cong \lim_I \mathrm{Hom}(\mathcal{F}_i, \phi_{i*}\, \mathcal{G})$$

with transition maps given by

$$\mathrm{Hom}(\mathcal{F}_j, \phi_{j*}\, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}_i, \phi_{i*}\, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}_i, (S_f \circ \phi_j)_* \mathcal{G}), \alpha_j \mapsto S_{f*}(\alpha_j) \circ \alpha_f$$

for every morphism $f : j \to i$ in I by inspection and the previous remark. Therefore, a morphism $L((\mathcal{F}_i, \alpha_f)) \to \mathcal{G}$ is given by a family $\alpha_i : \mathcal{F}_i \to \phi_{i*}\, \mathcal{G}$ such that

$$\begin{tikzcd}
\mathcal{F}_j \arrow[r, "\alpha_j"] \arrow[d, "\alpha_f"] & \phi_{j*}\, \mathcal{G} \arrow[d, equal] \\
S_{f*}\, \mathcal{F}_i \arrow[r, "S_{f*}(\alpha_i)"] & S_{f*} \phi_{i*}\, \mathcal{G}
\end{tikzcd}$$

commutes. This is the same as a morphism $(\mathcal{F}_i, \alpha_f) \to R\,\mathcal{G}$ in $\mathrm{PSh}(S_{(-)}, \mathrm{Set})$. We deduce the claim. □

Lemma 2.48. *Let $(\mathcal{F}_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$ and $(f_i : \phi_{i,P}^* \mathcal{F}_i \to \mathcal{F})_I$ be a colimit* *cocone in* $\mathrm{PSh}(\mathrm{Et}_S, \mathrm{Set})$*. Then, the family*

$$(\mathcal{F}_i \to \phi_{i*}\phi_{i,P}^*\mathcal{F}_i \overset{\phi_{i*}(f_i)}{\to} \phi_{i*}\mathcal{F})_I$$

with $\mathcal{F}_i \to \phi_{i*}\phi_{i,P}^*\mathcal{F}_i$ *the units of the adjunctions* $\phi_{i,P}^* \dashv \phi_{i*}$ *induces an isomorphism*

$$\mathcal{F} \overset{\sim}{\to} L((\mathcal{F}_i, \alpha_f))_I.$$

Proof. We obtain a sequence of natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_S,\mathrm{Set})}(L((\mathcal{F}_i,\alpha_f)),-) &\cong \lim_I \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_{S_i},\mathrm{Set})}(\mathcal{F}_i, \phi_{i*}(-))\\ &\cong \lim_I \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_{S_i},\mathrm{Set})}(\phi_{i,P}^*\mathcal{F}_i, -) \qquad \phi_{i,P}^* \dashv \phi_{i*}\\ &\cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_S,\mathrm{Set})}(\mathcal{F}, -)\end{aligned}$$

similar to the proof of the previous lemma. By Yoneda's Lemma, this natural transformation is given by precomposing with the isomorphism $L((\mathcal{F}_i, \alpha_f)) \to \mathcal{F}$ corresponding to $id_{\mathcal{F}}$. Along the sequence of isomorphism, this isomorphism is pointwise induced by the family

$$(\mathcal{F}_i(U_i) \to \phi_{i*}\phi_{i,P}^*\mathcal{F}_i(U_i) \overset{\phi_{i*}(f_i)}{\to} \phi_{i*}\mathcal{F}(U_i))_I$$

in question. □

Remark 2.49. We want to generalize the above lemma to sheaves and inverse images of sheaves. Recall the inverse image f^* to be given by the composition $(-)^{\#} \circ f_P^*$ for every morphism of schemes $f : X \to S$. In order to generalize the above, we need to prove that $\mathcal{F} \cong L((\mathcal{F}_i, \alpha_f))$ is a sheaf. To keep arguments simple we avoid introducing notation of sheaves and abelian group objects in $\mathrm{PSh}(S_{(-)}, \mathrm{Set})$. We would only need them for the proof of corollary 2.52 and theorem 2.58.

Lemma 2.50. *Let* $(\mathcal{F}_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$ *such that all* $\mathcal{F}_i$ *are sheaves. Then,* $L((\mathcal{F}_i, \alpha_f))$ *is a sheaf.*

Proof. Let

$$\{U_j \to U\}_{j \in J}$$

be a covering in Et_S^{fp}. We may assume that J is finite since Et_S^{fp} is noetherian. Then, there exists some i and a covering

$$\{U_{ij} \to U_i\}_{j \in J}$$

in $\mathrm{Et}^{fp}_{S_i}$ which base changes to

$$\{U_j \to U\}_{j\in J}$$

by proposition 2.30. Therefore, it suffices to prove $\phi_{i*}L((\mathcal{F}_i, \alpha_f))$ to be a sheaf for all $i \in I$. This was noticed in remark 2.46. □

Lemma 2.51. *Let* $(\mathcal{F}_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$ *be a presheaf on* $S_{(-)}$*. The family of morphisms* $\mathcal{F}_i \to \mathcal{F}_i^{\#}$ *induces a morphism* $(\mathcal{F}_i, \alpha_f) \to (\mathcal{F}_i^{\#}, \beta_f)$ *in* $\mathrm{PSh}(S_{(-)}, \mathrm{Set})$ *with* β_f *constructed below. The induced morphism*

$$L((\mathcal{F}_i, \alpha_f)) \to L((\mathcal{F}_i^{\#}, \beta_f))$$

is the sheafification of $L((\mathcal{F}_i, \alpha_f))$*.*

Proof. Let $f : i \to j$ be a morphism in I^{op}. We obtain a unique morphism

$$\beta_f : F_j^{\#} \to S_{f*}\mathcal{F}_i^{\#}$$

making the evident square commute. Therefore, $(\mathcal{F}_i^{\#})$ together with the morphisms β_f is indeed a presheaf on $S_{(-)}$ and the sheafification morphisms $\mathcal{F}_i \to \mathcal{F}_i^{\#}$ induce a morphism $(\mathcal{F}_i, \alpha_f) \to (\mathcal{F}_i^{\#})$ in $\mathrm{PSh}(S_{(-)}, \mathrm{Set})$. By the previous lemma, $L((\mathcal{F}_i^{\#}))$ is a sheaf. We derive the result by the sequence of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_S, \mathrm{Set})}(L((\mathcal{F}_i, \alpha_f)), \mathcal{G}) &\cong \lim_I \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_{S_i}, \mathrm{Set})}(\mathcal{F}_i, \phi_{i*}(\mathcal{G})) \\ &\cong \lim_I \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et}_{S_i}, \mathrm{Set})}(\mathcal{F}_i^{\#}, \phi_{i*}(\mathcal{G})) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et}_S, \mathrm{Set})}(L((\mathcal{F}_i^{\#}, \beta_f)), \mathcal{G}) \end{aligned}$$

for every $\mathcal{G} \in \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Set})$. □

Corollary 2.52. *Let* $(\mathcal{F}_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$ *such that all* $\mathcal{F}_i$ *are sheaves. Denote by*

$$(f_i : \phi_i^* \mathcal{F}_i \to \mathcal{F})_I$$

a colimit cocone in $\mathrm{Sh}(\mathrm{Et}_S, \mathrm{Set})$*. Then, the family*

$$(\mathcal{F}_i(U_i) \to \phi_{i*}\phi_i^* \mathcal{F}_i(U_i) \overset{\phi_{i*}(f_i)}{\to} \phi_{i*}\mathcal{F}(U_i))_I$$

with $\mathcal{F}_i \to \phi_{i*}\phi_i^*\mathcal{F}_i$ *the units of the adjunctions* $\phi_i^* \dashv \phi_{i*}$ *induces an isomorphism*

$$\mathcal{F} \xrightarrow{\sim} L((\mathcal{F}_i, \alpha_f)).$$

Proof. We have a sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_S,\mathrm{Set})}(L((\mathcal{F}_i,\alpha_f)),\mathcal{G}) &\cong \lim_I \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et}_{S_i},\mathrm{Set})}(\mathcal{F}_i,\phi_{i*}(\mathcal{G})) \\
&\cong \lim_I \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Et}_{S_i},\mathrm{Set})}(\phi_i^*\mathcal{F}_i,\mathcal{G}) \qquad \phi_i^* \dashv \phi_{i*} \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Et}_S,\mathrm{Set})}(\mathcal{F},\mathcal{G})
\end{aligned}$$

for every sheaf $\mathcal{G}$. We deduce the claim with similar arguments as in lemma 2.51. □

An important example is the following.

Example 2.53. Let $\mathcal{F}_0 \in \mathrm{Sh}(\mathrm{Et}_{S_0},\mathrm{Set})$ and define $\mathcal{F}$ resp. $\mathcal{F}_i$ to be the inverse images (of sheaves) of $\mathcal{F}_0$ under the structure morphism $S_i \to S_0$ resp. $S \to S_0$. Then, the family $(\mathcal{F}_i)_{i\in I}$ Together with the canonical family of morphisms

$$\alpha_f : \mathcal{F}_j \to S_{f*}\mathcal{F}_i$$

is an element of $\mathrm{PSh}(S_{(-)},\mathrm{Set})$ with all $\mathcal{F}_i$ sheaves. Observe

$$\phi_i^*\mathcal{F}_i \cong \mathcal{F}$$

to be canonically isomorphic. Let $U_i \in \mathrm{Et}_{S_i}^{fp}$ be an étale and finitely presentable S_i-scheme. Define $U = U_i \times_{S_i} S$ and $U_j = U_i \times_{S_i} S_j$ for every morphism $f : i \to j$ in I. Then, the cocone induced by units

$$(\mathcal{F}_j(U_j) \to \phi_{j*}\phi_j^*\mathcal{F}_j(U_j) \cong \mathcal{F}(U))_{i/I}$$

is a colimit cocone by the previous corollary.

Remark 2.54. The above is a generalization of the fully faithfulness in 2.30. Indeed, given some objects V_i and U_i of $\mathrm{Et}_{S_i}^{fp}$ define U_j, U, V_j and V as usual. Then, the family of units

$$- \times_{S_j} S : h_{U_j}(V_j) \to \phi_{j*}\phi_j^* h_{U_j}(V_j) = h_U(V)$$

induces a colimit diagram.

We want to extend example 2.53 to cohomology groups.

Remark 2.55. Let $(\mathcal{F}_i,\alpha_f)$ be a presheaf on $S_{(-)}$ such that all $\mathcal{F}_i$ are sheaves of abelian groups and all α_f are morphisms of abelian sheaves. Then, for every $U \in \mathrm{Et}_S^{fp}$ there exists a unique abelian group structure on $L((\mathcal{F}_i,\alpha_f))(U)$ making the colimit morphisms

$$\mathcal{F}_i(U_i) \to L((\mathcal{F}_i,\alpha_f))(U)$$

morphisms of abelian groups since $L((\mathcal{F}_i, \alpha_f))$ is pointwise a filterd colimit of abelian groups $\mathcal{F}_i(U_i)$. In particular, $L((\mathcal{F}_i, \alpha_f))$ is an abelian sheaf on Et_S by using lemma 2.21.

Lemma 2.56. *Let $(\mathcal{F}_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$ be a presheaf on $S_{(-)}$ such that all $\mathcal{F}_i$ are abelian sheaves and α_f are morphisms of abelian sheaves. If all $\mathcal{F}_i$ are flasque, then, $L((\mathcal{F}_i, \alpha_f))$ is flasque.*

Proof. Let $\mathcal{U} = \{U_j \to U\}_{j \in J}$ be a covering in Et_S^{fp}. We may assume J to be finite since Et_S^{fp} is noetherian. There exists some étale covering $\{U_{ij} \to U_i\}_{j \in J}$ in $\mathrm{Et}_{S_i}^{fp}$ which is isomorphic to $\mathcal{U}$ after base changing to S by proposition 2.30. We compute

$$\check{H}^q(\mathcal{U}, L((\mathcal{F}_i, \alpha_f))) \cong \underset{k \in i/I}{\mathrm{colim}} \check{H}^q(\{U_{ij} \times_{S_i} S_k \to U_i \times_{S_i} S_k\}_{j \in J}, \mathcal{F}_k)$$

since the covering is finite and filtered colimits commute with finite limits. Thus, all higher Čech cohomology groups vanish since all $\mathcal{F}_i$ are flasque and we deduce $L((\mathcal{F}_i, \alpha_f))$ to be flasque. □

Lemma 2.57. *Let $(\mathcal{F}_i, \alpha_f) \in \mathrm{PSh}(S_{(-)}, \mathrm{Set})$ be a presheaf on $S_{(-)}$ such that all $\mathcal{F}_i$ are abelian sheaves. There exists a morphism $(\mathcal{F}_i, \alpha_f) \to (\mathcal{G}_i, \beta_f)$ in $\mathrm{PSh}(S_{(-)}, \mathrm{Set})$ with*

1. *all $\mathcal{G}_i \in \mathrm{Sh}(S_i, \mathrm{Ab})$ are injective abelian sheaves.*
2. *all $\mathcal{F}_i \to \mathcal{G}_i$ are monomorphisms of abelian sheaves.*

Proof. We choose for every i in I an inclusion $\tilde{\gamma}_i : \mathcal{F}_i \subset \tilde{\mathcal{G}}_i$ such that $\tilde{\mathcal{G}}_i$ is an injective object in $\mathrm{Sh}(\mathrm{Et}_{S_i}, \mathrm{Ab})$. To make this choice functorial we define

$$\mathcal{G}_i = \prod_{f : j \to i \in I^{op}/i} S_{f*} \tilde{\mathcal{G}}_j$$

for every i in I. We obtain canonical morphisms of sheaves of abelian groups

$$\beta_f : \mathcal{G}_j \to S_{f*} \mathcal{G}_i$$

such that $(\mathcal{G}_i, \beta_f)$ is a presheaf on $S_{(-)}$. Furthermore, we obtain a morphism

$$\gamma_i : \mathcal{F}_i \to \mathcal{G}_i$$

induced by the morphisms

$$S_{f*}(\tilde{\gamma}_j) \circ \alpha_f : \mathcal{F}_i \to S_{f*} \mathcal{F}_j \subset S_{f*} \tilde{\mathcal{G}}_j .$$

We can check those to induce a morphism $(\mathcal{F}_i,\alpha_f)\to(\mathcal{G}_i,\beta_f)$ in $\mathrm{PSh}(S_{(-)},\mathrm{Set})$. The monomorphism $\tilde{\gamma}_i$ factorizes canonically through γ_i for every $i\in I$. Thus, γ_i is a monomorphism. At last, we recall all direct image functors to preserve injective sheaves by lemma 1.32. Thus, the abelian sheaves $\mathcal{G}_i$ are injective. We deduce the claim. □

Theorem 2.58. *Let $(\mathcal{F}_i,\alpha_f)\in\mathrm{PSh}(S_{(-)},\mathrm{Set})$ be a presheaf on $S_{(-)}$ such that all $\mathcal{F}_i$ are abelian sheaves and α_f are morphisms of abelian sheaves. Denote by*

$$(f_i:\alpha_i^*\mathcal{F}_i\to\mathcal{F})_I$$

a colimit cocone in $\mathrm{Sh}(\mathrm{Et}_S,\mathrm{Ab})$. Let U_i be in $\mathrm{Et}_{S_i}^{fp}$ and define $U=U_i\times_{S_i}S$ and $U_j=U_i\times_{S_i}S_j$. Then,

$$\operatorname*{colim}_{i/I}\mathrm{H}^q_{et}(U_j,\mathcal{F}_j)\overset{\sim}{\to}\mathrm{H}^q_{et}(U,\mathcal{F})$$

are isomorpic induced by the family of morphisms

$$(\mathcal{F}_j\to\phi_{j*}\phi_j^*\mathcal{F}_j\overset{\phi_{j*}(f_j)}{\to}\phi_{j*}\mathcal{F})_{i/I}$$

with $\mathcal{F}_i\to\phi_{i*}\phi_i^*\mathcal{F}_i$ the units of the adjunctions $\phi_i^*\dashv\phi_{i*}$.

Proof. Without loss of generality we may replace i/I by I. We prove the claim by induction over q. In the case $q=0$, this was proved in corollary 2.52. Assume the claim holds for $q\geq 0$. Choose a morphism $(\mathcal{F}_i,\alpha_f)\to(\mathcal{G}_i,\beta_f)$ as in the previous lemma. Denote by $\mathcal{G}_i\to\mathcal{H}_i$ a cokerenel for every i. We check their universal property induce a family

$$\gamma_f:\mathcal{H}_j\to S_{f*}\mathcal{H}_i$$

for every $f:i\to j$ in I such that

$$(\mathcal{G}_i,\beta_f)\to(\mathcal{H}_i,\gamma_f)$$

is a morphism in $\mathrm{PSh}(S_{(-)},\mathrm{Set})$. Observe both

$$\mathrm{colim}_I\,\mathrm{H}^{q+1}_{et}(U_i,\mathcal{G}_i)\cong 0 \text{ and } \mathrm{H}^{q+1}_{et}(U,\mathrm{colim}_I\,\mathcal{G}_i)\cong 0$$

to be zero since each $\mathcal{G}_i$ is flasque as well as $\mathrm{colim}_I\,\mathcal{G}_i$ is flasque by lemma 2.56. We obtain a commutative latter

$$\begin{tikzcd}
\mathrm{colim}_I\,\mathrm{H}^q_{et}(U_i,\mathcal{G}_i) \arrow[r] \arrow[d] & \mathrm{colim}_I\,\mathrm{H}^q_{et}(U_i,\mathcal{H}_i) \arrow[r] \arrow[d] & \mathrm{colim}_I\,\mathrm{H}^{q+1}_{et}(U_i,\mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
\mathrm{H}^q_{et}(U,\mathrm{colim}_I\,\mathcal{G}_i) \arrow[r] & \mathrm{H}^q_{et}(U,\mathrm{colim}_I\,\mathcal{H}_i) \arrow[r] & \mathrm{H}^{q+1}_{et}(U,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

by the long exact sequence of cohomology groups. Observe the upper row to be exact since filtered colimits are exact. The first two vertical morphisms are isomorphisms by induction hypothesis. We deduce the second vertical morphism to be an isomorphism by the four lemma. □

Example 2.53 generalizes to cohomology groups.

Example 2.59. Let $\mathcal{F}_0 \in \mathrm{Sh}(\mathrm{Et}_{S_0}, \mathrm{Ab})$ be a sheaf on Et_{S_0} and U_i be an element of $\mathrm{Et}^{fp}_{S_i}$. Define $\mathcal{F}$, $\mathcal{F}_i$, U_j and U as in example 2.53. Then, the cocone induced by units

$$(\mathrm{H}^q_{et}(U_j, \mathcal{F}_j) \to \mathrm{H}^q_{et}(U, \mathcal{F}))_{i/I}$$

is a colimit cocone.

3 Henselian rings and étale stalks

3.1 Points, neighbourhoods and stalks

In contrast to the Zariski site, the étale site of a field may not be trivial. However, the étale site of a field k consists only of finite disjoint unions of k iff k is separably closed. In that case, a sheaf on $\mathrm{Et}_{\mathrm{Spec}(k)}$ is completely determined by its value at $\mathrm{Spec}(k)$, i.e. $\Gamma(\mathrm{Spec}(k), -)$ defines an equivalence of categories. We are tempted to think of separably closed fields as the "true" points with respect to the étale topology.

Definition 3.1. Let X be a scheme and $x \in X$ be a point of X. A *geometric point at x* is a morphism $\mathrm{Spec}(k) \to X$ of schemes with k a separably closed field such that its topological image is x. In other words, a geometric point at x is a choice of an embedding of the residue field at x into a separably closed field.

Given a point $x \in X$ and an embedding of the residue field at x into a separable closure $\kappa(x) \subset \bar{k}^{sep}$, we denote by $\bar{x}$ the corresponding morphism

$$\mathrm{Spec}(\bar{k}^{sep}) \to \mathrm{Spec}(\kappa(x)) \to X.$$

We define the étale stalk analogously to the Zariski stalk.

Definition 3.2. Let X be a scheme and

$$p : \mathrm{Spec}(k) \to X$$

be a geometric point at $x \in X$. An *etale neighbourhood of p*, denoted by

$$f : (U, u) \to (X, p),$$

is an étale morphism $f : U \to X$ together with a geometric point $u : \mathrm{Spec}(k) \to U$ such that $p = f \circ u$. A morphism $f : (U, u) \to (V, v)$ of étale neighbourhoods of p is a morphism $f : U \to V$ of X-schemes such that $f \circ u = v$. We denote by NEt_p the category of étale neighbourhoods of p with canonical identity and composition.

Remark 3.3. Let k be a separably closed field and $p : \mathrm{Spec}(k) \to X$ be a geometric point with topological image x. Choose a factorization $p = e \circ \bar{x}$, i.e. choose a separable closure $\kappa(x) \subset \overline{\kappa(x)}^{sep}$ and an embedding $e : \overline{\kappa(x)}^{sep} \subset k$. Then, precomposing with e induces an equivalence

$$\mathrm{NEt}_{\bar{x}} \to \mathrm{NEt}_p\,.$$

Thus, at the level of neighbourhoods, the choice of a geometric point does only depend on its topological image.

Definition 3.4. Let X be a scheme, $\mathcal{F}$ be an object in $\mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Set})$ (resp. an object in $\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Set})$) and $p : \mathrm{Spec}(k) \to X$ be a geometric point at $x \in X$. Then, $\mathcal{F}$ induces a diagram

$$\mathrm{NEt}_p^{op} \to \mathrm{Set}, (U, u) \mapsto (F)(U), f \mapsto \mathcal{F}(f).$$

The *stalk of* $\mathcal{F}$ *at* p is the colimit of this diagram, denoted by

$$\mathcal{F}_p = \underset{(U,u)\to(X,p)}{\mathrm{colim}} \mathcal{F}(U)$$

By the previous remark, up to natural isomorphism two geometric points with the same topological image induce the same étale stalk.

Remark 3.5. Denote by $p : \mathrm{Spec}(k) \to X$ a geometric point.

1. We observe $\mathcal{F}_p \cong p^* \mathcal{F}(\mathrm{Spec}(k))$ to be canonically isomorphic by construction. In particular, taking stalks at p extends to an exact right adjoint

$$(-)_p : \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Set}) \to \mathrm{Set}$$

   such that $\Gamma(\mathrm{Spec}(k), -) \circ p^* \cong (-)_p$ are isomorphic since $\Gamma(\mathrm{Spec}(k), -)$ is an equivalence.

2. Let

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[ur, "q"] \arrow[r, "p"] & S \end{tikzcd}$$

   be a commutative diagram of schemes. The natural isomorphism $q^* \circ f^* \cong p^*$ induces a natural isomorphism $(f^* \mathcal{F})_q \cong \mathcal{F}_p$.

Remark 3.6. Let X be a scheme and $x \in X$. Whenever we write $\bar{x}$ we implicitely made a choice of an embedding of $\kappa(x)$ into some separable closure. In the context of stalks, remark 3.3 ensures the independence of that choice up to natural isomorphism. Furthermore, by "A property (P) holds on stalks." we indicate that property (P) holds after applying $(-)_{\bar{x}}$ for all $x \in X$.

Let us collect a number of results similar to ordinary sheaf theory on topological spaces.

Lemma 3.7. *Let* $\mathcal{F} \in \mathrm{PSh}(\mathrm{Et}_X, \mathrm{Set})$ *be a presheaf. The unit morphism* $\mathcal{F} \to \mathcal{F}^{\#}$ *is an isomorphim on stalks.*

Proof. Proposition 5.3.1 in [2]. □

Lemma 3.8. *A morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}(\mathrm{Et_S}, \mathrm{Set})$ is an isomorphism iff it is on stalks.*

Proof. The only if part is clear. The if part is Lemma 5.3.2 in [2]. □

Remark 3.9. Let p be a geometric point at some scheme X. Taking stalks preserves algebraic structures since it preserves finite limits. Thus, we obtain a unique factorization

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et_X}, \mathcal{C}) \arrow[r, "forget"] \arrow[d, dashed, "(-)_p"] & \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set}) \arrow[d, "(-)_p"] \\
\mathcal{C} \arrow[r, "forget"] & \mathrm{Set}
\end{tikzcd}$$

for $\mathcal{C} = \mathrm{Ab}, {}_\Lambda\mathrm{Mod}, \mathrm{CRing}$ and $forget$ the respective forgetful functor.

Corollary 3.10. *Let X be a scheme. A sequence*

$$0 \to A \to B \to C \to 0$$

in the category of sheaves on Et_X with values in abelian groups or Λ-modules is exact iff it is on stalks.

Proof. Proposition 5.3.3 [2] □

Example 3.11. Let X be a scheme and p be a geometric point at $x \in X$. The étale sheaf $\mathcal{O}_{X,et}$ represented by $\underline{\mathrm{Spec}}_X(\mathcal{O}_X[t])$ takes values in commutative rings. We denote by $\mathcal{O}_{X,p}$ the *etale stalk of $\mathcal{O}_{X,et}$ at p*. Since the étale stalk is defined as a filtered colimit, $\mathcal{O}_{X,p}$ inherits a canonical structure as a commutative ring.

Recall the Zariski stalk of $\mathcal{O}_X$ at some $x \in X$ to be a local ring. We ask what kind of structure $\mathcal{O}_{X,\bar{x}}$ possesses.

Example 3.12. For $X = \mathrm{Spec}(k)$ the spectrum of a field, $\mathcal{O}_{X,\bar{x}} \cong K$ is the separable closure of k chosen in $\bar{x} : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$. Indeed, by using every étale X scheme to be a finite disjoint union of spectra of finite and separable field extensions of k, we easily deduce

$$\mathcal{O}_{X,\bar{x}} \cong \operatorname*{colim}_{k \subset L \subset K} L = K$$

where the colimit is over all finite and separable field extension $k \subset L$ contained in K.

We will frequently use the following technical lemma.

Lemma 3.13. *Let X be a scheme, $U \subset X$ an open subset and $x \in X$. Then, the canonical morphism $\mathcal{O}_{U,\bar{x}} \to \mathcal{O}_{X,\bar{x}}$ is an isomorphism. Furthermore, the subdiagram of $\mathrm{NEt}_{\bar{x}}$ consisting of étale X-schemes which are affine schemes is cofinal.*

Proof. Observe that $\bar{x}$ canonically factorizes through $U \subset X$. Furthermore, we can check the canonical $\mathcal{O}_{U,\bar{x}} \cong ((U \subset X)^* \mathcal{O}_{X,et})_{\bar{x}} \to \mathcal{O}_{X,\bar{x}}$ to be an isomorphism by using 2. of remark 3.5.

The second claim is clear by using that every geometric point in a scheme factorizes through some affine open. □

Lemma 3.14. *Let X be a scheme and $x \in X$. The étale stalk $\mathcal{O}_{X,\bar{x}}$ is a local ring. Furthermore, the inclusion $\mathrm{Zar}_X \subset \mathrm{Et}_X$ induces a local and faithfully flat morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\bar{x}}$.*

Proof. First, recall that every local and flat morphism is faithfully flat. Let

$$f : (U, \bar{u}) \to (X, \bar{x})$$

be an étale neighbourhood of $\bar{x}$. Denote by u the topological image of $\bar{u}$. Then, every Zariski neighbourhood $V \subset U$ of u is canonically an étale neighbourhood

$$f : (V, \bar{u}) \to (X, \bar{x})$$

of $\bar{x}$. Thus, the canonical $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) \to U$ factorizes through the Zariski stalk

$$\mathrm{Spec}(\mathcal{O}_{U,u}) \to U$$

at u. Checking

$$(f_u : \mathcal{O}_{U,u} \to \mathcal{O}_{X,\bar{x}})_{(U,\bar{u}) \in \mathrm{NEt}_{\bar{x}}}$$

to be a filtered colimit cocone with transition morphisms induced by the Zariski stalk is then straight forward. Observe those transition morphisms to be local and flat by using that étale morphisms are flat. Thus, $\mathcal{O}_{X,\bar{x}}$ is local and all f_u are local and flat since the colimit is filtered. In particular, $f_x : \mathcal{O}_{X,x} \to \mathcal{O}_{X,\bar{x}}$ is local and flat, hence, faithfully flat. □

Here is a crucial observation.

Lemma 3.15. *Let X be a scheme and $x \in X$. Every standard étale morphism*

$$\mathcal{O}_{X,\bar{x}} \to (\mathcal{O}_{X,\bar{x}}[t]/(f))_g$$

such that there exists some prime ideal over the maximal ideal of $\mathcal{O}_{X,\bar{s}}$ admits a split.

Proof. Let $\mathcal{O}_{X,\bar{x}} \to (\mathcal{O}_{X,\bar{x}}[t]/(f))_g$ be a standard étale ring morphism. By lemma 3.13, we may assume that $X = \mathrm{Spec}(A)$ is affine and write

$$(f_B : B \to \mathcal{O}_{X,\bar{x}})_I$$

as a filtered colimit of étale ring morphisms $A \to B$. There exists some $A \to B$ étale such that all (finitely many) coefficience of g and f are elements of B and such that the derivative of f is invertible in $(B[t]/(f))_g$ by the very structure of filtered colimits of rings. We obtain a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathcal{O}_{X,\bar{x}} \arrow[d] \\ (B[t]/(f))_g \arrow[r] & (\mathcal{O}_{X,\bar{x}}[t]/(f))_g \end{tikzcd} .$$

We can check the induced $A \to (B[t]/(f))_g$ to extend to an étale neighbourhood of $\bar{x}$ such that the canonical $B \to (B[t]/(f))_g$ is a morphism of neighbourhoods. Then, we obtain a lift

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathcal{O}_{X,\bar{x}} \arrow[d] \\ (B[t]/(f))_g \arrow[r] \arrow[ur, dashed] & (\mathcal{O}_{X,\bar{x}}[t]/(f))_g \end{tikzcd}$$

making the evident triangles commute since $\mathcal{O}_{X,\bar{x}}$ is the colimit of all étale neighbourhoods of $\bar{x}$. In particular, the lift is a B-algebra morphism. Thus, it corresponds to an element $a \in \mathcal{O}_{X,\bar{x}}$ such that $f(a) = 0$ and $g(a) \in \mathcal{O}_{X,\bar{x}}^{\times}$. In particular, a induces a split

$$(\mathcal{O}_{X,\bar{x}}[t]/(f))_g \to \mathcal{O}_{X,\bar{x}}, t \mapsto a$$

of $\mathcal{O}_{X,\bar{x}} \to (\mathcal{O}_{X,\bar{x}}[t]/(f))_g$. □

Corollary 3.16. *Let X be a scheme and $x \in X$. The residue field of $\mathcal{O}_{X,\bar{x}}$ at its closed point is the separable closure K of $\kappa(x)$ chosen in the geometric point $\bar{x} : \mathrm{Spec}(K) \to X$.*

Proof. Denote by k the residue field of $\mathcal{O}_{X,\bar{x}}$ at its closed point. Because $\bar{x}$ factorizes through the induced $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) \to X$, we obtain a factorization $\kappa(x) \subset k \subset K$. It suffices to prove that every separable $0 \neq \bar{f} \in k[t]$ has a root in k. Recall that an element in $\mathcal{O}_{X,\bar{x}}$ is a unit iff it is a unit in k since $\mathcal{O}_{X,\bar{x}}$ is local. In particular, $\bar{f}$ lifts to a monic polynomial $f \in \mathcal{O}_{X,\bar{x}}[t]$. Furthermore, $\bar{f}$ and its derivative $\bar{f}'$ jointly generate $k[t]$. Thus, f and its derivative f' jointly generate $\mathcal{O}_{X,\bar{x}}[t]$. In particular,

$$\mathcal{O}_{X,\bar{x}} \to (\mathcal{O}_{X,\bar{x}}[t]/(f))_{f'}$$

is standard étale and admits a split by the previous lemma. The split corresponds to an $a \in \mathcal{O}_{X,\bar{x}}$ such that $f(a) = 0$ and $f'(a) \neq 0$. In particular, the reduction of a in k is a root of $\bar{f}$. Hence, $k \subset K$ is an isomorphism. □

3.2 Interlude on henselian rings

The results of this section mainly rest on results in commutative algebra. We will, therefore, mostly give references of its proofs.

Definition 3.17 ((Strictly) henselian)**.** Let (A, m, κ) be a local ring with maximal ideal m and residue field κ. Then,

1. A is *henselian* if for every monic polynomial $f \in A[t]$ and $a_0 \in \kappa$ such that $f(a_0) = 0$ and $f'(a_0) \neq 0$ in κ there exists $a \in A$ with $f(a) = 0$ in A which reduces to a_0.
2. A is *strictly henselian* if A is henselian and κ is separably closed.

Example 3.18. The étale stalk of $\mathcal{O}_{X,et}$ at some geometric point is strictly henselian by lemma 3.15 and corollary 3.16. Indeed, denote by $(\mathcal{O}_{X,\bar{x}}, m, k)$ the étale stalk with maximal ideal m and residue field k at m. Let $f \in \mathcal{O}_{X,\bar{x}}[t]$ be a monic polynomial and $a_0 \in k$ such that $f(a_0) = 0$ and $f'(a_0) \neq 0$ in k. Then, f, f' are coprime in $k[t]$. Since $\mathcal{O}_{X,\bar{x}}$ is local, f, f' are coprime in $\mathcal{O}_{X,\bar{x}}[t]$. Thus, the induced morphism

$$\mathcal{O}_{X,\bar{x}} \to \mathcal{O}_{X,\bar{x}}[t]_{f'}/(f)$$

is standard étale, hence, admits a split by lemma 3.15. Observe that a split corresponds to an $a \in \mathcal{O}_{X,\bar{x}}$ with $f(a) = 0$ in $\mathcal{O}_{X,\bar{x}}$ which reduces to a_0.

The étale stalk even carries a universal property.

Definition 3.19 (Strict henselianization)**.** Let (A, m) be a local ring and fix a separable closure $A/m \subset k$. The *category of strictly henselian A-algebras* has objects strictly henselian rings (B, n) together with a local morphism $(A, m) \to (B, n)$ and a morphism $k \subset B/n$ making the induced diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[d] & & B \arrow[d] \\
A/m \arrow[r, phantom, "\subset"] & k \arrow[r, phantom, "\subset"] & B/n
\end{tikzcd}$$

commute. Morphisms are A-algebra morphism such that the induced morphism of residue fields is a k-morphism. The *strict henselianization*

$$(A, m) \to (A^{sh}, m^{sh})$$

of (A, m) *with respect to* $A/m \subset k$ is an initial object in that category.

Remark 3.20. Up to equivalence, the above definition does not depend on the choice of a separable closure since the separable closure of some field is unique up to isomorphism. Therefore, the strict henselianization does not depend on the choice of a separable closure of A/m up to isomorphism. We will occasionally omit the embedding into a separable closure.

We will prove in a moment that $\mathcal{O}_{X,\bar{x}}$ is indeed the strict henselianization of $\mathcal{O}_{X,x}$. To do so we need a couple of equivalent characterizations of henselian rings.

Theorem 3.21. *Let* (A, m) *be a local ring. The following are equivalent.*

1. *A is henselian.*

2. *For any étale ring map* $f : A \to B$ *and* $n \in \operatorname{Spec}(B)$ *lying over* m *such that* f *induces an isomorphism* $\kappa(m) \cong \kappa(n)$, *there exists a section* g *of* f *with* $n = g^{-1}(m)$.

3. *Every finite A-algebra B decomposes into a finite product* $B \cong \prod B_i$ *of local rings* B_i. *In particular, each* $A \to B_i$ *is finite and local.*

Proof. This is (8) and (10) in [10, Tag 04GG] □

From this we can derive an equivalent characterization of being strictly henselian.

Proposition 3.22. *Let* (A, m) *be a local ring. The following are equivalent*

1. *A is strictly henselian.*

2. *For every étale morphism* $f : X \to \operatorname{Spec}(A)$ *and* $x \in X$ *lying over* m, *there exists a section* $g : \operatorname{Spec}(A) \to X$ *of* f *such that* $g(m) = x$.

Proof. Proposition 2.8.14 in [2]. □

Corollary 3.23. *Let* (A, m) *be a (strictly) henselian local ring and* $f : (A, m) \to (B, n)$ *be a finite and local morphism of local rings. Then, B is (strictly) henselian.*

Proof. We deduce by 3. of theorem 3.21 that B is henselian if A is since the composition of finite morphisms is finite. Assume A to be strictly henselian. Since f is finite, the induced field extension $A/m \subset B/n$ is finite. Thus, B/n is separably closed since A/m is. □

Corollary 3.24. *Let (A, m) be a strictly henselian local ring. Then, the quotient map*

$$A \to A/m$$

defines a geometric point $\bar{m}$ and the trivial étale neighbourhood is initial in $\mathrm{NEt}_{\bar{m}}$. In particular, the canonical

$$\Gamma(\mathrm{Spec}(A), -) \to (-)_{\bar{m}}$$

is an isomorphism.

Proof. Because A/m is separably closed, the first part is clear. Every étale neighbourhood of $\bar{m}$ has a point above m by definition. Therefore, every étale neighbourhood of $\bar{m}$ is split by 2. of proposition 3.22. This proves the claim. ☐

Proposition 3.25. *Let X be a scheme and $x \in X$. Then, $\mathcal{O}_{X,\bar{x}}$ is the strict henselianization of $\mathcal{O}_{X,x}$ with respect to the choice $\kappa(x) \subset k$ made in $\bar{x} : \mathrm{Spec}(k) \to X$. In particular, the strict henselianization of a local ring exists.*

Proof. [10, Tag 04HX] ☐

There exists also a notion of henselianization.

Definition 3.26. Let (A, m) be a local ring. The *category of henselian A-algebras* is the subcategory of A-algebras consisting of henselian local A-algebras together with local morphisms. The *henselianization of* (A, m) is an initial object in the category of henselian A-algebras.

Lemma 3.27. *Let (A, m) be a local ring. Then, the henselianization of (A, m) exists.*

Proof. Proposition 2.8.9 in [2] ☐

Here is an important property which the strict henselianization of a local ring inherits.

Proposition 3.28. *Let (A, m) be a local and noetherian ring. Then, its strict henselianization is noetherian.*

Proof. This is (iv) of Proposition 2.8.17 in [2] ☐

3.3 Finite morphisms

We can use our results to study direct image functors of finite morphisms.

Remark 3.29. Let $\bar{s} : \mathrm{Spec}(k) \to S$ be a geometric point and $f : X \to S$ be a finite morphism of schemes. Consider the induced cartesian diagram

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(\mathcal{O}_{S,\bar{s}}) \arrow[r] & S
\end{tikzcd}.$$

We have almost proved the canonical morphism

$$(f_* \mathcal{F})_{\bar{s}} \to g^* \mathcal{F}(X_{\bar{s}})$$

to be an isomorphism for every sheaf $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C})$ and $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}, {}_\Lambda\mathrm{Mod}$ in chapter 2. However, we will postpone its proof to example 6.19 because the construction fits nicely into the general framework of base change maps.

Theorem 3.30. *Let $f : X \to S$ be a finite morphism of schemes, $s \in S$ be a point and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C})$ be a sheaf for $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}, {}_\Lambda\mathrm{Mod}$. Let*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "n"] \arrow[rr, bend left, "h"] & X_{\bar{s}} \arrow[r, "g"] \arrow[d, "l"] & X \arrow[d, "f"] \\
\mathrm{Spec}(k) \arrow[r] \arrow[rr, bend right, "\bar{s}"'] & \mathrm{Spec}(\mathcal{O}_{S,\bar{s}}) \arrow[r] & S
\end{tikzcd}$$

be the diagram induced by the canonical $\mathrm{Spec}(\mathcal{O}_{S,\bar{s}}) \to S$ and quotient map $\mathcal{O}_{S,\bar{s}} \to k$ with each square cartesian. Then, there exists a natural isomorphism

$$(f_*\mathcal{F})_{\bar{s}} \xrightarrow{\sim} \prod_{x \in X_0} \mathcal{F}_{f \circ \bar{x}}$$

constructed in the proof. In particular, f_* is exact.

Proof. We may assume $S = \mathrm{Spec}(A)$ to be affine by using lemma 3.13. Then, $X = \mathrm{Spec}(B)$ is affine since f is affine. Thus, the pullback $X_{\bar{s}} = \mathrm{Spec}(C)$ is affine. Observe, the morphism

$$l : \mathcal{O}_{S,\bar{s}} \to C$$

to be finite since f is. In particular, we obtain an isomorphism

$$C \cong \prod_{i=1}^{n} C_i$$

of $\mathcal{O}_{S,\bar{s}}$-algebras for some finite, local and strictly henselian $\mathcal{O}_{S,\bar{s}}$-algebras (C_i, m_i) by 3. of theorem 3.21 combined with corollary 3.23. Observe the projection maps $C \to C_i$ to be localizations at some idempotents e_i. Thus, the induced

$$\{r_i : \mathrm{Spec}(C_i) \to X_{\bar{s}}\}_{i=1,\dots,n}$$

is an étale covering. Denote by $m \subset \mathcal{O}_{S,\bar{s}}$ the unique maximal ideal. We obtain a sequence of natural isomorphisms

$$\begin{aligned}
(f_* \mathcal{F})_{\bar{s}} &\cong g^* \mathcal{F}(X_{\bar{s}}) && \text{example 6.19} \\
&\cong \prod_{i=1}^{n} g^* \mathcal{F}(\mathrm{Spec}(C_i)) && \{\mathrm{Spec}(C_i) \to X_{\bar{s}}\} \text{ étale covering, } \mathcal{F} \text{ sheaf} \\
&\cong \prod_{i=1}^{n} \mathcal{F}_{\bar{m}_i} && C_i \text{ strictly henselian, corollary 3.24.}
\end{aligned}$$

The m_i correspond to the maximal ideals of C which in turn correspond to the prime ideals in C lying over m. In particular, the canonical $X_0 \to X_{\bar{s}} = \mathrm{Spec}(C)$ has topological image the maximal ideals of C since $X_0 \cong \mathrm{Spec}(C/mC)$ are canonically isomorphic. This proves the first claim.

We can check exactness of f_* stalkwise by lemma 3.8. We deduce f_* to be exact by the above natural isomorphism since all stalk functors as well as taking finite products are exact. □

3.4 Purely inseparable morphisms

With the above results on hand, we can now prove that certain morphisms of schemes induce equivalences of categories of étale sheaves.

Definition 3.31. Let $f : X \to S$ be a finite morphism of schemes. We call f *purely inseparable* if one of the following equivalent conditions hold.

1. For every $s \in S$ there exists exactly one $x \in X$ over s and the residue field extensions is purely inseparable.
2. Over each geometric point $\mathrm{Spec}(k) \to S$ with k algebraically closed there lies a unique geometric point $\mathrm{Spec}(k) \to X$ <.

Example 3.32. Let X be a scheme and X_{red} the reduction of X. Then, the canonical $X_{red} \to X$ is purely inseparable.

Remark 3.33. Being finite and 2. of the above definition are stable under base change. Therefore, purely inseparable is stable under base change.

Lemma 3.34. *Let $f : X \to S$ be purely inseparable. Let $\bar{s}$ be a geometric point at $s \in S$ and*

$$\begin{tikzcd}
X_0 \arrow[r, "l"] \arrow[d, "n"] & X \arrow[d, "f"] \\
\mathrm{Spec}(k) \arrow[r] & S
\end{tikzcd}$$

be a cartesian square of schemes with k an algebraically closed field. Then, the underlying topological space of X_0 consists of a single point.

Proof. We observe n to be purely inseparable by remark 3.33. Since the underlying topological space of $\mathrm{Spec}(k)$ is triviel we deduce the claim by 1. of the above definition. □

Lemma 3.35. *Let $f : X \to S$ be a purely inseparable morphism and $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}, {}_{\Lambda}\mathrm{Mod}$. Then, f_* and f^* define an equivalence of categories*

$$f_* : \mathrm{Sh}(\mathrm{Et_X}, \mathcal{C}) \simeq \mathrm{Sh}(\mathrm{Et_S}, \mathcal{C}) : f^*$$

Proof. Let $\mu : 1 \Rightarrow f_*f^*$ denote the unit and $\epsilon : f^*f_* \Rightarrow 1$ denote the counit. We can check stalkwise if those are isomorphisms by lemma 3.8. Let $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_S}, \mathcal{C})$ and $\mathcal{G} \in \mathrm{Sh}(\mathrm{Et_X}, \mathcal{C})$ be sheaves and $\bar{s}$ be a geometric point at S. We may enlarge k without changing the stalk by remark 3.3. Therefore, we may assume k to be algebraically closed. Then, there exists a unique factorization

$$\bar{s} : \mathrm{Spec}(k) \overset{\bar{x}}{\to} X \overset{f}{\to} S$$

by assumption. In particular, the morphism $(\mu_{\mathcal{F}})_{\bar{s}}$ factorizes as

$$\mathcal{F}_{\bar{s}} \cong (f^* \mathcal{F})_{\bar{x}} \overset{(f^*\mu_{\mathcal{F}})_{\bar{x}}}{\to} (f^*f_*f^* \mathcal{F})_{\bar{x}} \cong (f_*f^* \mathcal{F})_{\bar{s}}.$$

We recall $\epsilon_{f^* \mathcal{F}}$ to be a split of $f^*\mu_{\mathcal{F}}$ by the triangle conditions for adjoint functors. Therefore, it suffices to proof that ϵ is an isomorphism. Let

$$\bar{x}' : \mathrm{Spec}(k') \to X$$

be a geometric point at X. Again, we may assume k' to be algebraically closed. The fibre of f along $f \circ \bar{x}'$ has a trivial underlying topological space by lemma 3.34. Therefore, we obtain an isomorphism

$$(f^*f_* \mathcal{F})_{\bar{x}} \cong f_* \mathcal{F}_{f \circ \bar{x}'} \to \mathcal{F}_{\bar{x}'}$$

by theorem 3.30. By construction, this morphism is given by $(\epsilon_{\mathcal{F}})_{\bar{x}'}$. Therefore, $\epsilon_{\mathcal{F}}$ is stalkwise an isomorphism. We deduce the claim. □

Corollary 3.36. *Let $f : X \to S$ be a purely inseparable morphism of schemes. Then, the unit and counit of the adjunction $f^* \dashv f_*$ induce isomorphisms*

$$H^q(X, \mathcal{F}) \cong H^q(Y, f_* \mathcal{F})$$

and

$$H^q(Y, \mathcal{G}) \cong H^q(X, f^* \mathcal{G})$$

4 Artin Approximation

We now briefly present a famous result that goes back to Michael Artin. The original source is [14]. We are dealing with the following question.

Let A be a noetherian ring, $m \subset A$ be a proper ideal and $\hat{A}$ its m-adic completion. Let

$$F :_A \mathrm{Alg} \to \mathrm{Set}$$

be a finitely accessible, i.e. filtered colimits preserving, functor, c an integer and $\hat{a} \in F(\hat{A})$. Does there exist some $a \in F(A)$ such that a and $\hat{a}$ are equal in $F(A/m^c)$?

A lot of interesting geometric structures are classified by such functors.

Example 4.1. Let X be an Spec(A)-scheme for some commutative ring A. The functor mapping an A-algebra B to the set of isomorphism classes of finite étale $X \otimes_A B$-schemes with transition maps given by the base change of schemes is finitely accessible. This is a consequence of corollary 2.30.

Example 4.2. Let X be a scheme. Then, $\mathrm{Pic}(- \otimes_A X) :_B \mathrm{Alg} \to \mathrm{Set}$ is a finitely accessible functor with transition maps given by the base change of quasi-coherent sheaves. This is essentially (2) of [10, Tag 0B8W] and (2) and (3) of [10, Tag 01ZR]

Let $F = h_B$ be a representable functor with

$$B = A[t_1, .., t_k]/(f_1, ..., f_l)$$

a finitely presentable A-algebra. Observe that an element $f \in h_B(C)$ for some A-algebra C corresponds to elements $c_1, ..., c_k \in C$ with $f_i((c_j)) = 0$ for all i. Thus, for $\mathcal{F} = h_B$ the above question translates into the following.

Let $\hat{a} \in \hat{A}^k$ with $f_i(\hat{a}) = 0$ and c be an integer. Does there exist $a \in A^k$ such that $f_i(a) = 0$ and $a_j = \hat{a}_j \mod m^c$ for all i, j?

The crucial insight of Artin is that the above question has a positive answer for certain rings.

Theorem 4.3. *Let A be the henselianization at some prime ideal of a finite-type algebra over a field or an excellent discrete valuation ring. Let $m \subset A$ be a proper ideal and $\hat{A}$ be the m-adic completian of A. Then, for every elements $f_1, ..., f_l \in A[t_1, ..., t_k]$, c an integer and $\hat{a} \in \bar{A}^k$ such that $f_i(\hat{a}) = 0$ for all i, there exists an element $a \in A^k$ such that $f_i(a) = 0$ and $a_j = \hat{a}_j \mod m^c$ for all i, j.*

Proof. Theorem 1.10 in [14]. □

Combined with the above considerations we prove the following.

Theorem 4.4. *Let A be the henselianization at some prime ideal of a finite-type algebra over a field or an excellent discrete valuation ring. Let $m \subset A$ be a proper ideal and $\hat{A}$ be the m-adic completian of A. Let*

$$F :_A \mathrm{Alg} \to \mathrm{Set}$$

be a finitely accessible functor and $\hat{a} \in F(\hat{A})$ and c an integer. Then, there exists $a \in F(A)$ such that $\hat{z}$ and z become equal in $F(A/m^c)$.

Proof. We deduce the claim for functors representable by some finitely presentable A-algebra by using the above considerations. Recall every A-algebra to be a filtered colimit of finitely presentable A-algebras. By using that F is finitely accessible and the structure of filtered colimits of sets, we deduce the existence of some morphism of A-algebras $\hat{f} : B \to \hat{A}$ with B finitely presentable such that there exists some $b \in F(B)$ with $F(\hat{f})(b) = \hat{a}$. Observe $\hat{f}$ to be an element of $h_B(\hat{A})$. Thus, there exists $f \in F(A)$ such that f and $\hat{f}$ agree in $F(A/m^c)$. This is equivalent to the existence of a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\hat{f}"] \arrow[d, "f"] & \hat{A} \arrow[d] \\
A \arrow[r] & A/m^c
\end{tikzcd}$$

with $A \to A/m^c$ and $\hat{A} \to A/m^c$ the canonical maps. Applying F, we obtain a commutative diagram

$$\begin{tikzcd}
F(B) \arrow[r, "F(\hat{f})"] \arrow[d, "F(f)"] & F(\hat{A}) \arrow[d] \\
F(A) \arrow[r] & F(A/m^c)
\end{tikzcd} .$$

Since $\hat{a}$ is in the preimage of $F(\hat{f})$, we deduce the claim. ☐

Let us recall Grothendieck's existence theorem.

Definition 4.5. Let X be a noetherian scheme and $I \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Define $\mathrm{Coh}(X, I)$ to be the full subcategory of inverse systems of coherent $\mathcal{O}_X$-modules of those inverse systems $(\mathcal{F}_i)$ such that each $\mathcal{F}_i$ is annihilated by I^i and the transition maps induce isomorphisms $\mathcal{F}_i / I^{i+1} \mathcal{F}_{i+1} \to \mathcal{F}_i$.

Theorem 4.6 (Grothendieck's existence theorem)**.** *Let A be a noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes and define $\mathcal{I} = I\, \mathcal{O}_X$. Then, the assignment*

$$\mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(X, \mathcal{I}), \mathcal{F} \mapsto (\mathcal{F} / \mathcal{I}^i \mathcal{F})$$

defines an equivalence of categories.

Proof. [10, Tag088C] □

Here is an important application of Grothendieck's existence theorem and theorem 4.4.

Definition 4.7. Let X be a scheme. Denote by $FEt_X \subset \mathrm{Et}_X$ the full subcategory of finite étale maps.

Theorem 4.8. *Let*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

be a cartesian diagram of schemes with p proper and (A, m, k) henselian. Then, the base change to X_0 induces an equivalence of categories

$$- \times_X X_0 : \mathrm{FEt}_X \overset{\sim}{\to} \mathrm{FEt}_{X_0}$$

Proof. Theorem 3.1 in [14]. □

5 Calculations of étale cohomology groups

5.1 The relation between torsors and H^1

Recall the first Čech cohomology group for topological spaces to be given by "gluing data" for torsors. Its proof generalizes verbatim to sites and a generalized notion of torsor. Combined with the natural isomorphism of the first étale and the first Čech cohomology group, we will deduce the first étale cohomology group to classify torsors.

Definition 5.1. Let $\mathcal{G}$ be a sheaf of groups on a site T. A *$\mathcal{G}$-torsor* is a sheaf of sets $\mathcal{F}$ on T endowed with an action

$$\mathcal{G} \times \mathcal{F} \to \mathcal{F}$$

satisfying the following two conditions.

1. For all $X \in T$ there exists a covering of $\{U_i \to X\}_{i \in I}$ such that $\mathcal{F}(U_i) \neq \emptyset$ is non-empty.
2. If $\mathcal{F}(X) \neq \emptyset$ is non-empty, then, the action $\alpha(U) : \mathcal{G}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ is simply transitive.

The category of $\mathcal{G}$-torsors, denoted by $\mathrm{Tors}(T, \mathcal{G})$, is the full subcategory of $\mathcal{G}$-sheaves consisting of $\mathcal{G}$-torsors. Denote by $\mathrm{Tors}^{\cong}(T, \mathcal{G})$ the class of $\mathcal{G}$-torsors modulo isomorphism. This is a set.

Remark 5.2. Let $\mathcal{G}$ be a sheaf of groups on a site T. A $\mathcal{G}$-sheaf $\mathcal{F}$ is a $\mathcal{G}$-torsor iff for all $X \in \mathcal{C}$ there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic as $\mathcal{G}|_{U_i}$-sheaves to $\mathcal{G}|_{U_i}$ acting on itself by left multiplication.

Theorem 5.3. *Let T be a site, $\mathcal{G}$ be an abelian sheaf on T and $X \in T$ be a terminal object. Then, there is a bijection*

$$\mathrm{Tors}^{\cong}(T, \mathcal{G}) \xrightarrow{\sim} \check{H}^1(X, \mathcal{G})$$

where the assignement is as follows. Given some $\mathcal{G}$-torsor $\mathcal{F}$, we pick a covering

$$\{U_i \to X\}_{i \in I}$$

such that $\mathcal{F}(U_i) \neq \emptyset$ for all $i \in I$. Then, given an element $(s_i) \in \prod_I \mathcal{F}(U_i)$, there exists a unique

$$(\alpha_{ij}) \in \prod_{I^2} \mathcal{G}(U_i \times_X U_j)$$

such that

$$s_i|_{U_i \times_X U_j} = \alpha_{ij} s_j|_{U_i \times_X U_j}.$$

We assign $\mathcal{F}$ *the image of* (α_{ij}) *under the canonical*

$$\check{C}^1(\{U_i \to X\}_{i\in I}, \mathcal{G}) \to \check{H}^1(X, \mathcal{G}).$$

Proof. Theorem 3.38 in [15] □

Corollary 5.4. *Let* T *be a site and* $\mathcal{G}$ *be an abelian sheaf on* T *and* $X \in T$ *be a terminal object. Then,*

$$\mathrm{Tors}^{\cong}(T, \mathcal{G}) \xrightarrow{\sim} \mathrm{H}^1(X, \mathcal{G})$$

are isomorphic.

Proof. We compose the isomorphism of the previous lemma with the isomorphism $\check{H}^1(X, \mathcal{G}) \to \mathrm{H}^1(X, \mathcal{G})$ given in corollary 1.44. □

Lemma 5.5. *Let* X *be a noetherian and separated scheme. An étale sheaf* $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Set})$ *is representable by an étale* X*-scheme which is noetherian and separated iff*

1. *all stalks of* $\mathcal{F}$ *are finite sets and*
2. *for all* $a, b \in \mathcal{F}(U)$ *the set* $\{x \in U | a_{|x} \neq b_{|x}\} \subset U$ *is open.*

Proof. Lemma 3.18 in [3]. □

We can apply the above lemma in the context of étale cohomology.

Corollary 5.6. *Let* X *be a noetherian and separated scheme and* G *be a finite abelian group. Then, every* $\underline{G}$*-torsor* $\mathcal{F}$ *is representable by a finite étale* X*-scheme.*

Proof. Recall $\underline{G}$ to be representd by the étale scheme

$$\sqcup_G X \to X$$

by using 4. of example 2.11 and that G is finite. Then, $\mathcal{F}$ is locally isomorphic to $\underline{G}$ by remark 5.2. Therefore, we can easily check the conditions of 5.5 to be satisfied. Thus, combined with the fully faithfulness of the Yoneda embedding we deduce $\mathcal{F}$ to be representable by some étale X-scheme U which is étale-locally isomorphic to $\sqcup_G X \to X$. It remains to prove that U is a finite X-scheme. We may check the claim Zariski-locally on X. Thus, we may assume that $X = \mathrm{Spec}(A)$ is affine. Then, there exists a finite (since X is quasi-compact), étale and affine covering $\{\mathrm{Spec}(B_i) \to X\}_{i=1,\ldots,n}$ such that each base change of $U \to X$ to $\mathrm{Spec}(B_i)$ is finite. In particular, its base change along the canonical, faithfully flat morphism

$$\mathrm{Spec}(\prod_{i=1}^{n} B_i) \to X$$

is finite. Thus, $U \to X$ is finite. We deduce the claim. □

Remark 5.7. The bijection in 5.3 is functorial in the following sense. Let $f : X \to S$ be a morphism of schemes, $\mathcal{G} \in \mathrm{Sh}(\mathrm{Et}_S, \mathrm{Ab})$ be a sheaf and $\mathcal{F}$ be a $\mathcal{G}$-torsor with α the $\mathcal{G}$-sheaf structure. The inverse image of the underlying sheaf of sets $f^* \mathcal{F}$ inherits a canonical $f^* \mathcal{G}$-torsor structure induced by $f^*(\alpha)$ since f^* preserves finite limits. We obtain a morphism

$$f^* : \mathrm{Tors}^{\cong}(\mathrm{Et}_S, \mathcal{G}) \to \mathrm{Tors}^{\cong}(\mathrm{Et}_X, f^* \mathcal{G}), \mathcal{F} \mapsto f^* \mathcal{F}.$$

We can check that the diagram

$$\begin{tikzcd}
\mathrm{Tors}^{\cong}(\mathrm{Et}_S, \mathcal{G}) \arrow[r, "\sim", "5.3"'] \arrow[d, "f^*"] & \check{H}^1(S, \mathcal{G}) \arrow[d] \\
\mathrm{Tors}^{\cong}(\mathrm{Et}_X, f^* \mathcal{G}) \arrow[r, "\sim", "5.3"'] & \check{H}^1(X, f^* \mathcal{G})
\end{tikzcd}$$

commutes where the second vertical morphism is induced by the morphisms

$$\check{C}^\bullet(\{U_i \to S\}, \mu_{\mathcal{F}}) : \check{C}^\bullet(\{U_i \to S\}, \mathcal{F}) \to \check{C}^\bullet(\{U_i \to S\}, f_* f^* \mathcal{F}) = \check{C}^\bullet(\{U_i \times_S X \to X\}, f^* \mathcal{F})$$

for $\mu : 1 \Rightarrow f_* f^*$ the unit of the adjunction $f^* \dashv f_*$. Combined with remark 2.18, we deduce the diagram

$$\begin{tikzcd}
\mathrm{Tors}^{\cong}(\mathrm{Et}_S, \mathcal{G}) \arrow[r, "\sim", "5.3"'] \arrow[d, "f^*"] & \mathrm{H}^1_{et}(S, \mathcal{G}) \arrow[d] \\
\mathrm{Tors}^{\cong}(\mathrm{Et}_X, f^* \mathcal{G}) \arrow[r, "\sim", "5.3"'] & \mathrm{H}^1_{et}(X, f^* \mathcal{G})
\end{tikzcd}$$

to commute with the right vertical morphism of construction 2.17.

5.2 Second étale cohomology group

Lemma 5.8 (Kummer sequence)**.** *Let $n \geq 1$ be invertible in X. Then,*

$$n : \mathcal{O}_{X,et}^{\times} \to \mathcal{O}_{X,et}^{\times}, s \mapsto s^n$$

is an epimorphism of sheaves. We observe the sheaf of n-th roots of unity $\mu_{n,X}$ *of example 2.11 to be its kernel. In particular, we obtain a short exact sequence of abelian sheaves*

$$0 \to \mu_{n,X} \to \mathcal{O}_{X,et}^{\times} \to \mathcal{O}_{X,et}^{\times} \to 0.$$

Proof. We prove surjectivity on stalks, i.e. we prove the induced map

$$n_{\bar{x}} : (\mathcal{O}_{X,et}^{\times})_{\bar{x}} \to (\mathcal{O}_{X,et}^{\times})_{\bar{x}}$$

to be surjective for every geometric point $\bar{x}$ at X. We can check that

$$(\mathcal{O}_{X,et}^{\times})_{\bar{x}} \cong (\mathcal{O}_{X,\bar{x}})^{\times}$$

are canonically isomorphic as abelian groups. Thus, we may equivalently prove that the canonical morphism

$$f : \mathcal{O}_{X,\bar{x}} \to \mathcal{O}_{X,\bar{x}}[t]/(t^n - a), b \mapsto b$$

splits for every $a \in \mathcal{O}_{X,\bar{x}}^{\times}$. We observe $n^{-1}t(nt^{n-1}) - (t^n - a) = a \in \mathcal{O}_{X,\bar{x}}[t]^{\times}$ to be a unit. Thus, $t^n - a$ and its derivative nt^{n-1} jointly generate $\mathcal{O}_{X,\bar{x}}[t]$. Hence, the above map is étale by 2. of example 2.3. Furthemore, there exists a prime ideal in $\mathcal{O}_{X,\bar{x}}[t]/(t^n - a)$ over the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ since f is finite and étale. In particular, f splits by proposition 3.22 since $\mathcal{O}_{X,\bar{x}}$ is strictly henselian. □

Proposition 5.9. *Assume* $n \geq 1$ *to be invertible in* X*. Then, the representing* X*-scheme* $\underline{\mathrm{Spec}}_X(\mathcal{O}_X[t]/(t^n - 1))$ *of* $\mu_{n,x}$ *is étale. Furthermore, if* X *is a scheme over some strictly henselian local ring* (A, m, k) *such that* n *is invertible in* A*, then, choosing a primitive n-th root of unity in* A *induces an isomorphism*

$$\mu_{n,X} \cong \underline{\mathbb{Z}/n}.$$

Proof. Similar to the proof of the prevous lemma we deduce $\mathrm{Spec}\,\mathcal{O}_X[t]/(t^n - 1)$ to be an étale X-scheme if n is invertible in X. Assume X to be an A-scheme. Observe $t^n - 1$ to be separable in $k[t]$ since $n \in k^{\times}$. We deduce that $t^n - 1 \in k[t]$ splits into linear factors since k is separably closed. Thus, $t^n - 1 \in A[t]$ splits into linear factors since A is henselian. We obtain a canonical isomorphism

$$\mathrm{Spec}(A[t]/(t^n - 1)) \cong \sqcup_{\mathbb{Z}/n} \mathrm{Spec}(A)$$

by choosing a primitive n-th root of unity in A. In particular, base changing induces an isomorphism

$$\underline{\mathrm{Spec}}_X\, \mathcal{O}_X[t]/(t^n-1) \cong \sqcup_{\mathbb{Z}/n} X.$$

We deduce $\mu_{n,X} \cong \underline{\mathbb{Z}/n}$ to be isomorphic as sheaves of abelian groups by using example 2.11. □

Similar, we can proof the exactness of the Artin-Schreier sequence.

Lemma 5.10 (Artin-Schreier sequence)**.** *Let p be a prime number with $p \cdot 1 = 0$ in X. Then, the morphism*

$$\mathcal{O}_{X,et} \to \mathcal{O}_{X,et}, s \mapsto s^p - s$$

is an epimorphism in abelian sheaves on Et_X*. Furthermore, its kernel is isomorphic to* $\underline{\mathbb{Z}/p}$ *and we obtain a short exact sequence*

$$0 \to \underline{\mathbb{Z}/p} \to \mathcal{O}_{X,et} \to \mathcal{O}_{X,et} \to 0.$$

Proof. Proposition 7.2.3 in [2] □

Using the Kummer and Artin sequence, we can get hands on the constant étale sheaf $\underline{\mathbb{Z}/n}$. Here is an important example extending a result from cohomology of quasi-coherent sheaves to étale cohomology.

Lemma 5.11. *Let X/k be of finite type with k some separably closed field. Assume that k has positive characteristic p. If X is proper over k, then,*

$$\mathrm{H}^q_{et}(X, \underline{\mathbb{Z}/p}) = 0$$

is zero for all $q > dimX$.

Proof. Theorem 7.2.11 in [2] □

5.3 Cohomology of points and curves

The étale site of the spectrum of a field k is given by disjoint unions of spectra of separably and finite field extension of k. Those are studied with Galois theory. Let us briefly outline how étale cohomology generalizes Galois cohomology.

Remark 5.12. Let K/k be a finite Galois extension of fields and $\mathcal{F}$ be a sheaf of sets on $\mathrm{Et}_{\mathrm{Spec}(k)}$. The canonical left-action of $\mathrm{Gal}(K/k)$ on K extends to a natural left-action on $\mathcal{F}(\mathrm{Spec}(K))$ by functoriality. Choose a separable closure K^{sep}/k of k. By passing to colimits, we obtain an action of $\mathrm{Gal}(K^{sep}/k)$ on

$$\mathrm{colim}_{k \subset K \subset K^{sep} \text{ Galois}}\, \mathcal{F}(K)$$

where the diagram is taken over all Galois extensions K of k contained in K^{sep}. The choice of a separable closure K^{sep}/k corresponds to a geometric point

$$\bar{s} : \mathrm{Spec}(K^{sep}) \to \mathrm{Spec}(k).$$

Furthermore, we obtain a canonical isomorphism

$$\mathrm{colim}_{k \subset K \subset K^{sep}} \mathcal{F}(K) \cong \mathcal{F}_{\bar{s}}$$

and, hence, an action of $\mathrm{Gal}(K^{sep}/k)$ on $\mathcal{F}_{\bar{s}}$. This action is continuous with respect to the discrete topology on $\mathcal{F}_{\bar{s}}$.

Theorem 5.13. *Let k be a field and choose a separable closure $k \subset K^{sep}$. This defines a geometric point $\bar{s} : \mathrm{Spec}(K^{sep}) \to \mathrm{Spec}(k)$. Then, the assignement $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$ of the previous remark defines an equivalence*

$$\mathrm{Sh}(\mathrm{Et}_{\mathrm{Spec}(k)}, \mathrm{Ab}) \simeq \mathrm{Gal}(K^{sep}/k) - \mathrm{Mod}$$

where $\mathrm{Gal}(K^{sep}/k) - \mathrm{Mod}$ denotes the category of discrete abelian groups equipped with a continuous left action of $\mathrm{Gal}(K^{sep}/k)$. Furthermore, along this isomorphism the functors $\Gamma(\mathrm{Spec}(k), -)$ and taking fixed points

$$(-)^{\mathrm{Gal}(K^{sep}/k)} : \mathrm{Gal}(K^{sep}/k) - \mathrm{Mod} \to \mathrm{Ab}$$

agree. Therefore, we have a natural isomorphism

$$R\Gamma(\mathrm{Spec}(k), \mathcal{F}) \cong R(\mathcal{F}_{\bar{s}})^{\mathrm{Gal}(K^{sep}/k)}$$

of étale cohomology of $\mathcal{F}$ and group cohmology of $\mathcal{F}_{\bar{s}}$.

Proof. Proposition 5.7.8 in [2] □

Here is an important theorem from Galois cohomology.

Theorem 5.14. *Let K/k be a field extension of transcendental degree 1 with k separably closed. Then, the group cohomology groups*

1. *$H^i(Gal(\overline{K}^{sep}/K), A) = 0$ are zero for all $i \geq 2$ and any torsion $Gal(\overline{K}^{sep}/K)$-module A.*

2. *$H^i(Gal(\overline{K}^{sep}/K), \overline{K}^{sep\times}) = 0$ for $i = 1$ or $i \geq 3$. Furthermore, if p is the characteristik of K, then, $H^i(Gal(\overline{K}^{sep}/K), \overline{K}^{sep\times})$ is a p-torsion group.*

Proof. Theorem 4.5.11 in [2]. ☐

Let X be a scheme of dimension less or equal one and of finite type over a separably closed field k. Then, X splits into the set of points $s \in S$ with $dim\overline{\{s\}} = 1$, denoted by $S \subset X$ and closed points $X - S$ (i.e. points $x \in X$ with $dim\overline{\{x\}} = 0$). We can control cohomology of both types of points:

1. The field extension $\kappa(x)/k$ is finite for every $x \in X - S$. Thus, $\kappa(x)$ is separably closed. Hence, $\Gamma(\mathrm{Spec}(\kappa(x)), -)$ is an equivalence and all higher étale cohomology groups of étale sheaves on $\mathrm{Spec}(\kappa(x))$ vanish.
2. The field extension $\kappa(s)/k$ is of transcendental degree 1 for every $s \in S$. Thus, we can apply theorem 5.14.

Let $\mathcal{F}$ be an abelian sheaf on Et_X. Define j to be the induced morphism

$$j : \mathrm{Spec}(\prod_S \mathcal{O}_{X,s}) \to X.$$

Observe that S is finite since X is noetherian. Thus,

$$\mathrm{Spec}(\prod_S \mathcal{O}_{X,s}) \cong \coprod_S \mathrm{Spec}(\mathcal{O}_{X,s})$$

are isomorphic. Furthermore, the dimensions

$$dim(\mathcal{O}_{X,s}) = 0$$

are zero for all $s \in S$ since $dim\overline{\{s\}} = dim X = 1$. We deduce that

$$\mathrm{Spec}(\mathcal{O}_{X,s}) = \{s\}$$

consists of a single element. Hence, j has topological image S. We relate $\mathrm{H}^q_{et}(X, \mathcal{F})$ to the étale cohomology of $\mathcal{F}$ at points of S by the unit

$$\mu_{\mathcal{F}} : \mathcal{F} \to j_* j^* \mathcal{F}$$

of the adjunction $j^* \dashv j_*$.

Definition 5.15. Let Y be a scheme and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_Y, \mathrm{Ab})$. We define the *support of* $\mathcal{F}$ to be

$$supp(\mathcal{F}) = \{y \in Y | \, \mathcal{F}_{\bar{y}} \neq 0\}.$$

Lemma 5.16. *With the above notation, the following holds.*

1. *The support of the kernel and of the cokernel of* $\mu_{\mathcal{F}} : \mathcal{F} \to j_* j^* \mathcal{F}$ *are contained in* $X - S$ *for every abelian sheaf* $\mathcal{F}$ *on* Et_X*.*
2. *The support of* $R^q j_* \mathcal{G}$ *is contained in* $X - S$ *for every* $\mathcal{G} \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{Spec}(\prod_S \mathcal{O}_{X,s})}, \mathrm{Ab})$ *and every* $q \geq 1$*.*

Proof. Lemma 7.2.5 in [2]. □

Here is a technical characterization classifying such sheaves.

Lemma 5.17. *Assume* Y *to be a noetherian scheme and* $\mathcal{G}$ *to be an étale sheaf of abelian groups on* Y*. Denote by* $T \subset Y$ *the set of (Zariski) closed points. Then, the following two conditions are equivalent:*

1. *supp*$(\mathcal{G}) \subset T$ *and*
2. *the canonical morphism* $\mathcal{G} \to \prod_T i_{t*} i_t^* \mathcal{G}$ *induces an isomorphism* $\mathcal{G} \cong \oplus_T i_{t*} i_t^* \mathcal{G}$ *for* $i_t : \mathrm{Spec}(\kappa(t)) \to Y$ *the closed immersions.*

Proof. Lemma 7.2.4 in [2]. □

Corollary 5.18. *In the situation of the previous lemma, if supp*$(\mathcal{F}) \subset T$*, then,* $\mathrm{H}^q(Y, \mathcal{F})$ *is zero for all* $q \geq 1$*.*

Proof. We obtain $\mathcal{G} \cong \oplus_T i_{t*} i_t^* \mathcal{G}$ by the previous lemma. Furthermore, every i_t is finite since $t \in T$ is closed and Y is noetherian. In particular, i_{t*} is exact by theorem 3.30. Thus, we obtain a canonical isomorphism

$$\mathrm{H}^q(Y, i_{t*} i_t^* \mathcal{G}) \cong \mathrm{H}^q(\mathrm{Spec}(\kappa(t)), i_t^* \mathcal{G}).$$

Recall sheaf cohomology to commute with finite direct sums. Combined, we obtain a sequence of isomorphisms

$$\mathrm{H}^q_{et}(Y, \mathcal{G}) \cong \mathrm{H}^q_{et}(Y, \oplus_T i_{t*} i_t^* \mathcal{G}) \cong \oplus_T \mathrm{H}^q_{et}(Y, i_{t*} i_t^* \mathcal{G}) \cong \oplus_T \mathrm{H}^q(\mathrm{Spec}(\kappa(t)), i_t^* \mathcal{G}).$$

We deduce $\kappa(t)$ to be separably closed since k is and all $\kappa(t)/k$ are finite field extensions. In particular,

$$\mathrm{H}^q(\mathrm{Spec}(\kappa(t)), i_t^* \mathcal{G}) \cong 0$$

is zero for all $t \in T$ and $q \geq 1$. We deduce the claim. □

Definition 5.19. Let S be a scheme and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Ab})$ be a sheaf. We call $\mathcal{F}$ a *torsion sheaf* if every stalk of $\mathcal{F}$ at a geometric point is a torsion abelian group. Because stalks commute with filtered colimits of sheaves, we easily deduce $\mathcal{F}$ to be a torsion sheaf iff

$$\mathcal{F} \cong \operatorname{colim}_{\mathbb{N}}(\ker(\mathcal{F} \xrightarrow{\cdot n} \mathcal{F}))$$

are canonically isomorphic.

Example 5.20. Every sheaf in the image of the canonical inclusion

$$\operatorname{Sh}(\operatorname{Et}_{S,\mathbb{Z}/n} \operatorname{Mod}) \subset \operatorname{Sh}(\operatorname{Et}_{\mathrm{S}}, \operatorname{Ab})$$

is a torsion sheaf for $n > 0$.

Combining all the above we can proof the following theorem.

Theorem 5.21. *Let X be a scheme of finite type over a separably closed field k with $dim X \leq 1$. Then,*

$$\mathrm{H}^q_{et}(X, \mathcal{F}) = 0$$

is zero for every torsion sheaf $\mathcal{F}$ on Et_X and $q > 2$.

Proof. We use the notation of lemma 5.16. Given an abelian sheaf $\mathcal{F}$ on X_{et}, the unit morphism $\mu_{\mathcal{F}} : \mathcal{F} \to j_* j^* \mathcal{F}$ induces exact sequences

$$0 \to \ker(\mu_{\mathcal{F}}) \to \mathcal{F} \to \operatorname{im}(\mu_{\mathcal{F}}) \to 0$$

$$0 \to \operatorname{im}(\mu_{\mathcal{F}}) \to j_* j^* \mathcal{F} \to \operatorname{coker}(\mu_{\mathcal{F}}) \to 0.$$

Then, $\mathrm{H}^q_{et}(X, \operatorname{coker}(\mu_{\mathcal{F}})) \cong 0$ and $\mathrm{H}^q_{et}(X, \ker(\mu_{\mathcal{F}})) \cong 0$ are zero for $q > 0$ by lemma 5.16 and corollary 5.18. We obtain an isomorphism

$$\mathrm{H}^q_{et}(X, \mathcal{F}) \cong \mathrm{H}^q_{et}(X, j_* j^* \mathcal{F})$$

for $q > 1$ by the long exact sequence in cohomology. Thus, we may replace $\mathcal{F}$ by $j_* j^* \mathcal{F}$. Consider the Leray spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p_{et}(X, R^q j_* j^* \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}_{et}(\operatorname{Spec}(\prod_S \mathcal{O}_{X,s}), j^* \mathcal{F}) = \mathrm{E}^{p+q}.$$

Recall, $R^q j_* j^* \mathcal{F}$ to be the sheaf associated to

$$U \mapsto \mathrm{H}^q_{et}(\operatorname{Spec}(\prod_S \mathcal{O}_{X,s}) \times_X U, s^* \mathcal{F})$$

by lemma 1.34. We may replace X with X_{red} since this does not change the cohomology groups by example 3.32 and corollary 3.36. Then, each $\mathcal{O}_{X,s}$ is reduced and noetherian with a single prime ideal. We deduce each $\mathcal{O}_{X,s}/k$ to be a transcendental field extension of degree 1 for $s \in S$. Thus, every étale morphism $U \to \mathrm{Spec}(\prod_S \mathcal{O}_{X,s}) = \sqcup_S \mathrm{Spec}(\mathcal{O}_{X,s})$ is given by a finite disjoint union of separable and finite field extensions of the fields $\kappa(s)$ for $s \in S$. Furthermore,

$$\{\mathrm{Spec}(\mathcal{O}_{X,s}) \to \mathrm{Spec}(\prod_{s\in S} \mathcal{O}_{X,s})\}_S$$

is an étale covering. We obtain an isomorphism

$$\mathrm{H}^q_{et}(\mathrm{Spec}(\prod_S \mathcal{O}_{X,s}), -) \cong \oplus_S \mathrm{H}^q_{et}(\mathrm{Spec}(\mathcal{O}_{X,s}), -)$$

by using the sheaf property. Combined with theorem 5.13 and theorem 5.14, we obtain

$$\mathrm{H}^q_{et}(\mathrm{Spec}(\prod_S \mathcal{O}_{X,s}) \times_X U, s^* \mathcal{F}) \cong 0$$

and

$$H^q(\mathrm{Spec}(\prod_S \mathcal{O}_{X,s}), s^* \mathcal{F}) \cong 0$$

are zero for all $q > 1$ and $U \in \mathrm{Et}_X$. At last, we obtain $\mathrm{H}^p_{et}(X, R^q j_* j^* \mathcal{F}) \cong 0$ to be zero for every $p, q > 0$ by lemma 5.16 and lemma 5.18. All together, we deduce that the edge morphism in the above spectral sequence yields an isomorphism

$$\mathrm{H}^p_{et}(X, j_* j^* \mathcal{F}) \cong \mathrm{H}^p_{et}(\mathrm{Spec}(\prod_S \mathcal{O}_{X,s}), s^* \mathcal{F}) \cong 0$$

for every $q > 2$. □

We deduce the following theorem in a similar way by applying 2. of theorem 5.14.

Theorem 5.22. *Let X be a reduced scheme over a separably closed field k of characteristic p with $dim X \leq 1$. Then, $\mathrm{H}^q_{et}(X, \mathcal{O}^\times_{X,et})$ are p-torsion groups for $q = 2, 3$ and zero for $q \geq 4$.*

Proof. Theorem 7.2.7 in [2]. □

Remark 5.23. We are tempted to use corollary 3.36 to get rid the assumption that X is reduced in the previous theorem. However there is a small subtlety. Let $p : X_{red} \to X$ be the canonical morphism. We obtain an isomorphism

$$\mathrm{H}^q_{et}(X, \mathcal{O}^\times_{X,et}) \cong \mathrm{H}^q_{et}(X_{red}, p^* \mathcal{O}^\times_{X,et})$$

by corollary 3.36. However, the canonical $p^* \mathcal{O}_{X,et}^{\times} \to \mathcal{O}_{X_{red},et}^{\times}$ is not an isomorphism in general.

We can combine our restults with the Kummer sequence in order to obtain the following corollary.

Corollary 5.24. *Let X be a reduced scheme over a separably closed field k of characteristic p with $dim X \leq 1$. Let $n \in \mathbb{N}$ invertible in k. Then, the connecting homomorphism*

$$\mathrm{H}^1_{et}(X, \mathcal{O}_{X,et}^{\times}) \to \mathrm{H}^2_{et}(X, \mu_{n,X})$$

in the long exact sequence associated to the Kummer sequence 5.8 is an epimorphism.

Proof. Theorem 7.2.9 in [2]. □

6 Proper base change theorem

The proper base change theorem in topology relates the stalks of the higher direct images to the higher cohomology groups of their fibres.

Theorem 6.1 (Proper base change in topology)**.** *Assume $f : X \to Y$ to be a proper (i.e. universally closed) morphism of topological spaces. Let $\mathcal{F}$ be a sheaf of sets on X and $y \in Y$. Denote by X_y the fibre of f over y and by $\mathcal{F}|_{X_y}$ the inverse image of $\mathcal{F}$ along the fibre X_y. Then, the canonical morphism*

$$f_* \mathcal{F}_y \xrightarrow{\sim} (\mathcal{F}|_{X_y})(X_y)$$

is an isomorphism. Furthermore, the canonical morphism

$$(R^q f_*(\mathcal{F}))_y \xrightarrow{\sim} \mathrm{H}^q(X_y, \mathcal{F}|_{X_y})$$

is also an isomorphism for $\mathcal{F}$ an abelian sheaf and $q \geq 0$.

Proof. Theorem 17.2 in [16] □

Let $f : X \to S$ be a morphism of schemes and $\mathcal{F}$ an abelian sheaf on Et_X. For every geometric point $\bar{s} : \mathrm{Spec}(k) \to S$ we can construct a canonical morphism

$$(R^q f_*(\mathcal{F}))_{\bar{s}} \xrightarrow{\sim} \mathrm{H}^q_{et}(X_{\bar{s}}, \mathcal{F}|_{S_{\bar{s}}}). \tag{1}$$

We want to prove that (1) is an isomorphism for f a proper morphism of schemes. To address this claim we generalize the morphism (1) in a suitable sense. We will exhibit many common morphisms to be instances of this general construction. The key insight is that (1) extends to a natural transformation

$$\begin{tikzcd}
\mathrm{D}^+(\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C})) \arrow[r, "|_{S_{\bar{s}}}"] \arrow[d, "Rf_*"] & \mathrm{D}^+(\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}_{\bar{s}}}, \mathcal{C})) \arrow[d, "R\Gamma(\mathrm{X}_{\bar{s}}\text{,-})"] \\
\mathrm{D}^+(\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C})) \arrow[r, "(\text{-})_{\bar{s}}"] \arrow[ur, Rightarrow] & \mathrm{D}^+(\mathrm{Ab})
\end{tikzcd} .$$

6.1 Generalized base change

Let

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[u, "L_2"] & \mathcal{A}_4 \arrow[u, "L_3"']
\end{tikzcd}$$

be a square of functors. Assume L_2 resp. L_3 to admit a right adjoint R_2 resp. R_3. Denote by μ_i resp. ϵ_i the unit resp. counit of the adjunction $L_i \dashv R_i$. We construct two morphisms by the following assignment.

$$\mathrm{Hom}_{\mathrm{Fun}}(L_3 \circ L_4, L_1 \circ L_2) \leftrightarrow \mathrm{Hom}_{\mathrm{Fun}}(L_4 \circ R_2, R_3 \circ L_1) \tag{2}$$

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\ \mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow] \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{A}_1 \arrow[r, equal] & \mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "R_3"] & \mathcal{A}_4 \\ \mathcal{A}_1 \arrow[u, equal] \arrow[r, "R_2"'] & \mathcal{A}_3 \arrow[u, "L_2"'] \arrow[ul, Rightarrow, "\epsilon_2"'] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow] \arrow[r, equal] & \mathcal{A}_4 \arrow[u, equal] \arrow[ul, Rightarrow, "\mu_3"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}_3 \arrow[r, "L_2"] \arrow[d, equal] & \mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[r, equal] \arrow[d, "R_3"] & \mathcal{A}_2 \arrow[d, equal] \\ \mathcal{A}_3 \arrow[r, equal] \arrow[ur, Rightarrow, "\mu_2"] & \mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow] & \mathcal{A}_4 \arrow[r, "L_3"] \arrow[ur, Rightarrow, "\epsilon_3"] & \mathcal{A}_2 \end{tikzcd} \leftarrowtail \begin{tikzcd} \mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[d, "R_3"] \\ \mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow] & \mathcal{A}_4 \end{tikzcd}$$

Similarly, let

$$\begin{tikzcd} \mathcal{A}_1 \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] \\ \mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"] \end{tikzcd}$$

be a square of functors such that R_1 resp. R_4 admits a left adjoint L_1 resp. L_4. Denote by μ_i resp. ϵ_i the unit resp. counit of the adjunction $L_i \dashv R_i$. Define two morphisms by the assignment

$$\mathrm{Hom}_{\mathrm{Fun}}(R_2 \circ R_1, R_4 \circ R_3) \leftrightarrow \mathrm{Hom}_{\mathrm{Fun}}(L_4 \circ R_2, R_3 \circ L_1) \tag{3}$$

$$\begin{tikzcd}
 & & & \mathcal{A}_1 \arrow[r, equal] \arrow[d, equal] \arrow[dr, Rightarrow, "\mu_1"] & \mathcal{A}_1 \arrow[d, "L_1"] \\
\mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] & \mapsto & \mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"] & & \mathcal{A}_3 \arrow[d, "L_4"'] \arrow[dr, Rightarrow, "\epsilon_4"] & \mathcal{A}_4 \arrow[l, "R_4"] \arrow[d, equal] \\
 & & & \mathcal{A}_4 \arrow[r, equal] & \mathcal{A}_4
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}_2 \arrow[r, equal] \arrow[d, "R_1"] & \mathcal{A}_2 \arrow[d, equal] & & & \\
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] \arrow[ur, Rightarrow, "\epsilon_1"] & \mathcal{A}_2 \arrow[d, "R_3"] & \mapsfrom & \mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[d, equal] \arrow[ur, Rightarrow] & \mathcal{A}_4 \arrow[d, "R_4"] & & \mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow] & \mathcal{A}_4 \\
\mathcal{A}_3 \arrow[r, equal] \arrow[ur, Rightarrow, "\mu_4"] & \mathcal{A}_3 & & &
\end{tikzcd}$$

Lemma 6.2. *The natural morphisms (2) and (3) are pairwise inverses. Furthermore, given a square of functors*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"]
\end{tikzcd}$$

such that each L_i admits a right adjoint R_i, we obtain an induced bijection

$$\mathrm{Hom}_{\mathrm{Fun}}(L_3 \circ L_4, L_1 \circ L_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}}(R_2 \circ R_1, R_4 \circ R_3) \tag{4}$$

Then, $\alpha \in \mathrm{Hom}_{\mathrm{Fun}}(L_3 \circ L_4, L_1 \circ L_2)$ corresponds to $\beta \in \mathrm{Hom}_{\mathrm{Fun}}(R_2 \circ R_1, R_4 \circ R_3)$ iff the induced diagram

$$\begin{tikzcd}
\mathrm{Hom}(L_3 \circ L_4 -, -) \arrow[r, "\sim", "\phi"'] & \mathrm{Hom}(-, R_4 \circ R_3 -) \\
\mathrm{Hom}(L_1 \circ L_2 -, -) \arrow[u, "\alpha^*"] \arrow[r, "\sim", "\psi"'] & \mathrm{Hom}(-, R_2 \circ R_1 -) \arrow[u, "\beta_*"]
\end{tikzcd}$$

commutes with the horizontal arrows induced by the adjunctions $L_3 \circ L_4 \dashv R_4 \circ R_3$ and $L_1 \circ L_2 \dashv R_2 \circ R_1$.

Proof. We only proof one direction of (2). The other parts follow with almost identical arguments. Let

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow]
\end{tikzcd}$$

be a natural transformation. The composition in question is given by

$$\begin{tikzcd}
 & & \mathcal{A}_2 \arrow[r, equal] \arrow[d, equal] & \mathcal{A}_2 \\
\mathcal{A}_1 \arrow[r, equal] \arrow[d] & \mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "R_3"'] & \mathcal{A}_4 \arrow[u, "L_3"'] \arrow[ul, Rightarrow, "\epsilon_3"] \arrow[d, equal] \\
\mathcal{A}_1 \arrow[r, "R_2"'] & \mathcal{A}_3 \arrow[u, "L_2"] \arrow[ul, Rightarrow, "\epsilon_2"] \arrow[r, "L_4"'] \arrow[d, equal] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow] \arrow[r, equal] & \mathcal{A}_4 \arrow[ul, Rightarrow, "\mu_3"] \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, equal] & \mathcal{A}_3 \arrow[ul, Rightarrow, "\mu_2"]
\end{tikzcd} .$$

We obtain equalities

$$\begin{tikzcd}
\mathcal{A}_1 \\
\mathcal{A}_3 \arrow[u, "L_2"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A}_1 \arrow[r, equal] \arrow[d, equal] & \mathcal{A}_1 \\
\mathcal{A}_1 \arrow[r, "R_2"'] & \mathcal{A}_3 \arrow[u, "L_2"] \arrow[ul, Rightarrow, "\epsilon_2"] \arrow[d, equal] \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, equal] & \mathcal{A}_3 \arrow[ul, Rightarrow, "\mu_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}_2 \arrow[r, equal] \arrow[d, equal] & \mathcal{A}_2 \\
\mathcal{A}_2 \arrow[r, "R_3"'] & \mathcal{A}_4 \arrow[u, "L_3"'] \arrow[ul, Rightarrow, "\epsilon_3"] \arrow[d, equal] \\
\mathcal{A}_4 \arrow[u, "L_3"] \arrow[r, equal] & \mathcal{A}_4 \arrow[ul, Rightarrow, "\mu_3"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A}_2 \\
\mathcal{A}_4 \arrow[u, "L_3"]
\end{tikzcd}$$

by the triangle conditions for adjoint functors. This proves

$$\begin{tikzcd}
 & & \mathcal{A}_2 \arrow[r, equal] \arrow[d, equal] & \mathcal{A}_2 \\
\mathcal{A}_1 \arrow[r, equal] \arrow[d, equal] & \mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "R_3"'] & \mathcal{A}_4 \arrow[u, "L_3"'] \arrow[ul, Rightarrow, "\epsilon_3"] \arrow[d, equal] \\
\mathcal{A}_1 \arrow[r, "R_2"'] & \mathcal{A}_3 \arrow[u, "L_2"] \arrow[ul, Rightarrow, "\epsilon_2"] \arrow[r, "L_4"'] \arrow[d, equal] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow] \arrow[r, equal] & \mathcal{A}_4 \arrow[ul, Rightarrow, "\mu_3"] \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, equal] & \mathcal{A}_3 \arrow[ul, Rightarrow, "\mu_2"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow]
\end{tikzcd}$$

to be equal. Let us prove the second claim. We convince ourselves that the bijection in

(4) is given by (2) applied to the diagram

$$\begin{tikzcd}
\mathcal{A}_2 \arrow[r, equal] & \mathcal{A}_2 \\
\mathcal{A}_1 \arrow[u, "L_1"] & \mathcal{A}_4 \arrow[u, "L_3"] \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, equal] & \mathcal{A}_3 \arrow[u, "L_4"'] \arrow[uul, Rightarrow]
\end{tikzcd}$$

resp. (3) applied to the diagram

$$\begin{tikzcd}
\mathcal{A}_3 \arrow[d, equal] \arrow[drr, Rightarrow] & \mathcal{A}_1 \arrow[l, "R_2"'] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, equal] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"] & \mathcal{A}_2 \arrow[l, "R_3"]
\end{tikzcd} .$$

By Yoneda's Lemma, the diagram in question commutes iff

$$\beta(A) = \phi \circ \alpha^* \circ \psi^{-1}(id_{R_2 \circ R_1(A)})$$

are equal for every $A \in \mathcal{A}_2$. A straightforward calculation proves the right hand side to be given by the image of α under (4) applied to A. The other direction follows by similar arguments. ☐

Example 6.3. Let

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"]
\end{tikzcd}$$

be a commutative (up to natural isomorphism) square of functors such that each L_i admits a right adjoint R_i. We obtain a natural isomorphism of the diagram of right adjoints corresponding to the commutativity under the bijection (4). Then, the natural transformation

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow] & \mathcal{A}_4
\end{tikzcd}$$

induced by (2) resp. by (3) are the same.

Example 6.4. Let $L : \mathcal{A} \to \mathcal{B}$ be a functor with right adjoint R. Then, the natural transformation corresponding to the identity

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "L"] \arrow[d, equal] \arrow[dr, equal] & \mathcal{B} \\
\mathcal{A} \arrow[r, equal] & \mathcal{A} \arrow[u, "L"]
\end{tikzcd}$$

yields the unit and to the identity

$$\begin{tikzcd}
\mathcal{B} \arrow[r, equal] \arrow[d, "R"] \arrow[dr, equal] & \mathcal{B} \arrow[d, equal] \\
\mathcal{A} & \mathcal{B} \arrow[l, "R"]
\end{tikzcd}$$

yields the counit.

Proposition 6.5. *Let*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "L_5"] & \mathcal{A}_5 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[r, "L_6"] \arrow[ul, Rightarrow] & \mathcal{A}_6 \arrow[u, "L_7"] \arrow[ul, Rightarrow]
\end{tikzcd}$$

be two natural transformations such that each L_2, L_3 *and* L_7 *admits a right adjoint* R_2, R_3 *and* R_7*. We obtain an induced natural transformation*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "L_5"] & \mathcal{A}_5 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[r, "L_6"'] & \mathcal{A}_6 \arrow[u, "L_7"] \arrow[ull, Rightarrow]
\end{tikzcd}.$$

Then, the composition

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[r, "L_5"] \arrow[d, "R_3"] & \mathcal{A}_5 \arrow[d, "R_7"] \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow] & \mathcal{A}_4 \arrow[r, "L_6"] \arrow[ur, Rightarrow] & \mathcal{A}_6
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[r, "L_5"] & \mathcal{A}_5 \arrow[d, "R_7"] \\
\mathcal{A}_3 \arrow[r, "L_4"'] \arrow[urr, Rightarrow] & \mathcal{A}_4 \arrow[r, "L_6"] & \mathcal{A}_6
\end{tikzcd}$$

corresponding under the bijection (2) to the upper natural transformations agree.

Proof. The composition in question is given by

$$\begin{tikzcd}
 & & \mathcal{A}_2 \arrow[r, equal] & \mathcal{A}_2 \arrow[r, "L_5"] & \mathcal{A}_5 \arrow[r, "R_7"] & \mathcal{A}_6 \\
\mathcal{A}_1 \arrow[r, equal] & \mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "R_3"] \arrow[u, equal] & \mathcal{A}_4 \arrow[r, "L_6"'] \arrow[u, "L_3"'] \arrow[ul, Rightarrow, "\epsilon_3"'] & \mathcal{A}_6 \arrow[r, equal] \arrow[u, "L_7"'] \arrow[ul, Rightarrow] & \mathcal{A}_6 \arrow[u, equal] \arrow[ul, Rightarrow, "\mu_7"'] \\
\mathcal{A}_1 \arrow[r, "R_2"'] \arrow[u, equal] & \mathcal{A}_3 \arrow[r, "L_4"'] \arrow[u, "L_2"'] \arrow[ul, Rightarrow, "\epsilon_2"'] & \mathcal{A}_4 \arrow[r, equal] \arrow[u, "L_3"'] \arrow[ul, Rightarrow] & \mathcal{A}_4 \arrow[u, equal] \arrow[ul, Rightarrow, "\mu_3"'] & &
\end{tikzcd} .$$

We need to prove that

$$\begin{tikzcd}
\mathcal{A}_2 \arrow[r, equal] & \mathcal{A}_2 \\
\mathcal{A}_2 \arrow[r, "R_3"] \arrow[u, equal] & \mathcal{A}_4 \arrow[u, "L_3"'] \arrow[ul, Rightarrow, "\epsilon_3"'] \\
\mathcal{A}_4 \arrow[r, equal] \arrow[u, "L_3"] & \mathcal{A}_4 \arrow[u] \arrow[ul, Rightarrow, "\mu_3"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}_2 \\
\mathcal{A}_4 \arrow[u, "L_3"]
\end{tikzcd}$$

agree. This is part of the triangle conditions for adjoint functors. ☐

Proposition 6.6. *Let*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[u, "L_2"] & \mathcal{A}_4 \arrow[u, "L_3"'] \arrow[ul, Rightarrow] \\
\mathcal{A}_5 \arrow[r, "L_7"] \arrow[u, "L_5"] & \mathcal{A}_6 \arrow[u, "L_6"'] \arrow[ul, Rightarrow]
\end{tikzcd}$$

be two natural transformations such that each L_2, L_3, L_5, L_6 *admit right adjoints* R_2, R_3, R_5, R_6*. We obtain an induced natural transformation*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] & \mathcal{A}_4 \arrow[u, "L_3"'] \\
\mathcal{A}_5 \arrow[r, "L_7"] \arrow[u, "L_5"] & \mathcal{A}_6 \arrow[u, "L_6"'] \arrow[uul, Rightarrow]
\end{tikzcd} .$$

Then, the composition

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"'] & \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[d, "R_5"'] \arrow[ur, Rightarrow] & \mathcal{A}_4 \arrow[d, "R_6"] \\
\mathcal{A}_5 \arrow[r, "L_7"'] \arrow[ur, Rightarrow] & \mathcal{A}_6
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"'] & \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[d, "R_5"'] & \mathcal{A}_4 \arrow[d, "R_6"] \\
\mathcal{A}_5 \arrow[r, "L_7"'] \arrow[uur, Rightarrow] & \mathcal{A}_6
\end{tikzcd}$$

corresponding under the bijection (2) to the upper natural transformations agree.

Proof. The composition in question is given by

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[rr, equal] \arrow[dd, equal] & & \mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "R_3"] & \mathcal{A}_4 \arrow[r, "R_6"] \arrow[d, equal] & \mathcal{A}_6 \arrow[dd, equal] \\
& \mathcal{A}_3 \arrow[r, equal] \arrow[d, equal] \arrow[ul, Rightarrow, "\epsilon_2"'] & \mathcal{A}_3 \arrow[r, "L_4"] \arrow[u, "L_2"] & \mathcal{A}_4 \arrow[r, equal] \arrow[u, "L_3"'] \arrow[ul, Rightarrow] & \mathcal{A}_4 \arrow[ul, Rightarrow, "\mu_3"'] & \\
\mathcal{A}_1 \arrow[r, "R_2"] & \mathcal{A}_3 \arrow[r, "R_5"] & \mathcal{A}_5 \arrow[r, "L_7"] \arrow[u, "L_5"] \arrow[ul, Rightarrow, "\epsilon_5"'] & \mathcal{A}_6 \arrow[rr, equal] \arrow[u, "L_6"'] \arrow[ul, Rightarrow] & & \mathcal{A}_6 \arrow[ul, Rightarrow, "\mu_6"']
\end{tikzcd} .$$

We need to prove that

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[rr, equal] \arrow[dd, equal] & & \mathcal{A}_1 & & \mathcal{A}_2 \arrow[r, "R_3"] & \mathcal{A}_4 \arrow[r, "R_6"] \arrow[d, equal] & \mathcal{A}_6 \arrow[dd, equal] \\
& \mathcal{A}_3 \arrow[r, equal] \arrow[d, equal] \arrow[ul, Rightarrow, "\epsilon_2"'] & \mathcal{A}_3 \arrow[u, "L_2"] & \text{resp.} & \mathcal{A}_4 \arrow[r, equal] \arrow[u, "L_3"] & \mathcal{A}_4 \arrow[ul, Rightarrow, "\mu_3"'] & \\
\mathcal{A}_1 \arrow[r, "R_2"] & \mathcal{A}_3 \arrow[r, "R_5"] & \mathcal{A}_5 \arrow[u, "L_5"] \arrow[ul, Rightarrow, "\epsilon_5"'] & & \mathcal{A}_6 \arrow[rr, equal] \arrow[u, "L_6"] & & \mathcal{A}_6 \arrow[ul, Rightarrow, "\mu_6"']
\end{tikzcd}$$

is the counit resp. unit of $L_2 \circ L_5 \dashv R_5 \circ R_2$ resp. $L_3 \circ L_6 \dashv R_6 \circ R_3$. This is well known. ☐

Remark 6.7. The corresponding claims for squares of right adjoints as in (3) is proved with almost identical arguments.

Corollary 6.8. *Let*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "L_5"] & \mathcal{A}_5 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow] \arrow[r, "L_6"] & \mathcal{A}_6 \arrow[u, "L_7"] \arrow[ul, Rightarrow]
\end{tikzcd}$$

be natural transformations as in proposition 6.5. We obtain three natural transformations

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[d, "R_3"] & \\
\mathcal{A}_3 \arrow[ur, Rightarrow] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[r, "L_6"] & \mathcal{A}_6
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \arrow[r, "L_5"] \arrow[d, "R_3"] & \mathcal{A}_5 \arrow[d, "R_7"] \\
& \mathcal{A}_4 \arrow[ur, Rightarrow] \arrow[r, "L_6"] & \mathcal{A}_6
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[r, "L_5"] & \mathcal{A}_5 \arrow[d, "R_7"] \\
\mathcal{A}_3 \arrow[urr, Rightarrow] \arrow[r, "L_4"'] & \mathcal{A}_4 \arrow[r, "L_6"] & \mathcal{A}_6
\end{tikzcd}$$

under the bijection (2). If two of them are isomorphisms, then, so is the third. The corresponding claim for the vertical composition

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"] & \mathcal{A}_4 \arrow[u, "L_3"] \arrow[ul, Rightarrow] \\
\mathcal{A}_5 \arrow[u, "L_5"] \arrow[r, "L_7"] & \mathcal{A}_6 \arrow[u, "L_6"] \arrow[ul, Rightarrow]
\end{tikzcd}$$

is also true.

Corollary 6.9. *Let*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"]
\end{tikzcd}$$

be a natural transformation such that R_2 *resp.* R_4 *admits a left adjoint* L_2 *resp.* L_4*. Let* $E : \mathcal{A}_4 \to \mathcal{A}_5$ *be an equivalence of categories. Then, its corresponding natural transformation under (2)*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[ur, Rightarrow] \arrow[r, "L_4"] & \mathcal{A}_4
\end{tikzcd}$$

is an isomorphism iff the natural transformation

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"'] & \mathcal{A}_2 \arrow[d, "E\circ R_3"] \\
\mathcal{A}_3 \arrow[r, "E\circ L_4"'] \arrow[ur, Rightarrow] & \mathcal{A}_5
\end{tikzcd}$$

corresponding to the outer square of

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[r, equal] \arrow[d, "R_3"] \arrow[dr, phantom, "\cong"] & \mathcal{A}_2 \arrow[d, "E\circ R_3"] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"] & \mathcal{A}_5 \arrow[l, "E^{-1}"]
\end{tikzcd}$$

is an isomorphism.

Proposition 6.10. *Let*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"]
\end{tikzcd}$$

be a morphism of functors such that R_2 *resp.* R_4 *admits a left adjoint* L_2 *resp.* L_4*. Denote by* μ_1 *the unit of the adjunction* $L_1 \dashv R_1$ *and by* ϵ_4 *the counit of the adjunction* $L_4 \dashv R_4$*. If two of the natural transformations*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, equal] \arrow[d, equal] \arrow[dr, Rightarrow, "\mu_1"] & \mathcal{A}_1 \arrow[d, "L_1"] \\
\mathcal{A}_1 \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[l, "R_1"] \\
\mathcal{A}_3 \arrow[d, "L_4"] & \\
\mathcal{A}_4 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{A}_1 \arrow[d, "L_1"] \\
& \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[d, "L_4"] \arrow[dr, Rightarrow, "\epsilon_4"] & \mathcal{A}_4 \arrow[l, "R_4"'] \arrow[d, equal] \\
\mathcal{A}_4 \arrow[r, equal] & \mathcal{A}_4
\end{tikzcd}$$

are isomorphisms, then, the third is an isomorphism iff the morphism corresponding to

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "R_2"] \arrow[dr, Rightarrow] & \mathcal{A}_2 \arrow[l, "R_1"] \arrow[d, "R_3"] \\
\mathcal{A}_3 & \mathcal{A}_4 \arrow[l, "R_4"]
\end{tikzcd}$$

under bijection (3) is an isomorphism. The corresponding claim for the case of left adjoints and bijection (2) is also true.

Proof. This is true by definition. ☐

Lemma 6.11. *Let $\alpha : \mathcal{A} \to \mathcal{B}$ be an exact functor and β a right adjoint of α. Assume $\mathcal{A}$ and $\mathcal{B}$ to have enough injective objects. Then, $\alpha : D^+ \mathcal{A} \to D^+ \mathcal{B}$ is a left adjoint of $R\beta : D^+ \mathcal{B} \to D^+ \mathcal{A}$.*

Proof. Denote by $K^+ \mathcal{A}$ resp. $K^+ \mathcal{B}$ the category of left bounded chain complexes modulo chain homotopy. Applying α resp. β pointwise induces an adjoint pair

$$\alpha : K^+ \mathcal{A} \leftrightarrow K^+ \mathcal{B} : \beta.$$

Recall the universal $l_{\mathcal{B}} : K^+ \mathcal{B} \to D^+ \mathcal{B}$ to admit a right adjoint $\mathcal{I} : D^+ \mathcal{B} \to K^+ \mathcal{B}$ which maps a complex to an injective resolution. Let $C \in K^+ \mathcal{A}$ and $D \in K^+ \mathcal{B}$ be bounded below chain complexes. We obtain a sequence of natural isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{D^+ \mathcal{B}}(\alpha C, D) &\cong \operatorname{Hom}_{D^+ \mathcal{B}}(\alpha C, \mathcal{I}(D)) && D \cong \mathcal{I}(D) \text{ in } D^+ \mathcal{B} \\
&\cong \operatorname{Hom}_{K^+ \mathcal{B}}(\alpha C, \mathcal{I}(D)) && l_{\mathcal{B}} \dashv \mathcal{I} \\
&\cong \operatorname{Hom}_{K^+ \mathcal{A}}(C, \beta \mathcal{I}(D)) && \alpha \dashv \beta \\
&\cong \operatorname{Hom}_{D^+ \mathcal{A}}(C, \beta \mathcal{I}(D)) && \beta \text{ preserves injective objects and is exact} \\
&\cong \operatorname{Hom}_{D^+ \mathcal{A}}(C, R\beta D) && \text{by construction of } R\beta
\end{aligned}$$

where we omitted $l_{\mathcal{B}}$ because it is the identity on objects. ☐

Remark 6.12. Let

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"'] \arrow[ur, Rightarrow] & \mathcal{A}_4 \arrow[u, "L_3"']
\end{tikzcd}$$

be a natural transformation. Assume all $\mathcal{A}_i$ to be abelian and L_2 resp. L_3 to admit right adjoints R_2 resp. R_3. Assume further that all $\mathcal{A}_i$ have enough injective objects and all L_i are exact. Then, all right derived functors of L_i are given by pointwise applying L_i. We obtain a natural transformation

$$\begin{tikzcd}
D^+ \mathcal{A}_1 \arrow[r, "L_1"] & D^+ \mathcal{A}_2 \\
D^+ \mathcal{A}_3 \arrow[u, "L_2"] \arrow[r, "L_4"'] \arrow[ur, Rightarrow] & D^+ \mathcal{A}_4 \arrow[u, "L_3"']
\end{tikzcd}.$$

By the previous lemma, $L_2 : D^+ \mathcal{A}_3 \to D^+ \mathcal{A}_1$ resp. $L_2 : D^+ \mathcal{A}_3 \to D^+ \mathcal{A}_1$ are left adjoint to RR_2 and RR_3. Therefore, the above square corresponds to a natural transformation

$$\begin{tikzcd}
\mathrm{D}^+ \mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "RR_2"] & \mathrm{D}^+ \mathcal{A}_2 \arrow[d, "RR_3"] \\
\mathrm{D}^+ \mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow, "\beta"] & \mathrm{D}^+ \mathcal{A}_4
\end{tikzcd}$$

under bijection (2). Here is an explicit calculation. Given a bounded below complex $K^\bullet$ in $\mathcal{A}_1$, choose quasi-isomorphisms

$$a : K^\bullet \to I^\bullet \text{ and } b : L_1 I^\bullet \to J^\bullet$$

with $I^\bullet$ and $J^\bullet$ complexes of injectve objects. Observe that

$$b \circ L_1(a) : L_1 K^\bullet \to L_1 I^\bullet \to J^\bullet$$

is a quasi-isomorphism since L_1 is exact. The natural transformation

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] \arrow[d, "R_2"] & \mathcal{A}_2 \arrow[d, "R_3"] \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{A}_4
\end{tikzcd}$$

corresponding to the natural transformation

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "L_1"] & \mathcal{A}_2 \\
\mathcal{A}_3 \arrow[r, "L_4"] \arrow[u, "L_2"] \arrow[ur, Rightarrow] & \mathcal{A}_4 \arrow[u, "L_3"']
\end{tikzcd}$$

induces a morphism $L_4 \circ R_2(I^\bullet) \to R_3 \circ L_1(I^\bullet)$ by pointwise application. At the level of derived categories we obtain a morphism

$$L_4 \circ \mathrm{R}\, R_2(K^\bullet) \xrightarrow{\sim} L_4 \circ R_2(I^\bullet) \to R_3 \circ L_1(I^\bullet) \overset{R_3(b)}{\to} R_3(J^\bullet) \overset{\sim}{\leftarrow} \mathrm{R}\, R_3 \circ L_1(K^\bullet).$$

We can check that this construction pointwise agrees with β. We call the natural transformation β the derived version of α.

Example 6.13. Let $f : X \to S$ be a morphism of schemes and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Set})$ be a sheaf. By the very construction, the natural transformation

$$\Gamma(\mathrm{S},\text{-}) \to \Gamma(X, f^*-) \cong \Gamma(S, f_* f^* -)$$

corresponding to

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et_S}, \mathcal{C}) \arrow[d, "\Gamma(\mathrm{S},\text{-})"] & \mathrm{Sh}(\mathrm{Et_X}, \mathcal{C}) \arrow[l, "f_*"'] \arrow[d, "\Gamma(\mathrm{X},\text{-})"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is given by the unit of the adjunction $f^* \dashv f_*$. Furthermore, for $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_S}, {}_\Lambda\mathrm{Mod})$ the derived version induces a morphism

$$\mathrm{H}^q(S, \mathcal{F}) \to \mathrm{H}^q(X, f^* \mathcal{F})$$

which agrees with the morphism of construction 2.17.

With the developed compatibility results on hand we can prove remark 2.18.

Corollary 6.14. *Let $f : X \to S$ be a morphism of schemes and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Ab})$ be an abelian sheaf. Denote by $\mu : 1 \Rightarrow f_* f^*$ the unit of the adjunction $f^* \dashv f_*$ and by*

$$\iota_S : \mathrm{PSh}(\mathrm{Et}_S, \mathrm{Ab}) \subset \mathrm{Sh}(\mathrm{Et_S}, \mathrm{Ab}) \;\; \textit{and} \;\; \iota_X : \mathrm{PSh}(\mathrm{Et}_X, \mathrm{Ab}) \subset \mathrm{Sh}(\mathrm{Et_X}, \mathrm{Ab})$$

the respective inclusions. Then, the diagram

$$\begin{tikzcd}
\check{H}^1(S, \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\alpha"] & \mathrm{H}^1_{et}(S, \mathcal{F}) \arrow[d, "\beta"] \\
\check{H}^1(X, f^* \mathcal{F}) \arrow[r, "\sim"] & \mathrm{H}^1_{et}(X, f^* \mathcal{F})
\end{tikzcd}$$

with vertical isomorphisms of corollary 1.44, β the morphism of construction 2.17 and α induced by the family of morphisms

$$\check{C}^\bullet(\{U_i \to S\}, \mu_{\mathcal{F}}) : \check{C}^\bullet(\{U_i \to S\}, \mathcal{F}) \to \check{C}^\bullet(\{U_i \to S\}, f_* f^* \mathcal{F}) \cong \check{C}^\bullet(\{U_i \times_S X \to X\}, f^* \mathcal{F})$$

commutes.

Proof. For the sake of clarity, we give only a sketch of the proof and leave some claims to the reader. We consider the following diagram of functors

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[rrr, "f^*"] \arrow[dd, "\iota_S"] \arrow[dr] & & & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[dd, "\iota_X"] \arrow[dl] \\
& D^+ \mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f^*"] \arrow[dd, "R\iota_S"] & D^+ \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[dd, "R\iota_X"] & \\
\mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[rrr, "f_P^*"] \arrow[dr] & & & \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[dl] \\
& D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[d, "R\check{H}^0(S,-)"] & D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "R\check{H}^0(X,-)"] & \\
& D^+ \mathrm{Ab} \arrow[r, equal] \arrow[d, "\mathrm{H}^1(-)"] & D^+ \mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] & \\
& \mathrm{Ab} \arrow[r, equal] & \mathrm{Ab} &
\end{tikzcd}$$

with oblique arrows the canonical ones. Observe the top and bottom of the “cube” to be commutative. We convince ourselves that α is given by

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f^*"] \arrow[d, "\iota_S"] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "\iota_X"] & \\
\mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[ur, Rightarrow] \arrow[dr] & \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & \\
& D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[d, "R\check{H}^0(S,-)"] & D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "R\check{H}^0(X,-)"] \\
& D^+ \mathrm{Ab} \arrow[r, equal] \arrow[ur, Rightarrow] \arrow[d, "\mathrm{H}^1(-)"] & D^+ \mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] \\
& \mathrm{Ab} \arrow[r, equal] & \mathrm{Ab}
\end{tikzcd}$$

and β is given by

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[dr] & & \\
& D^+\,\mathrm{Sh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[r, "f^*"] \arrow[d, "R\iota_S"] & D^+\,\mathrm{Sh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[d, "R\iota_X"] \\
& D^+\,\mathrm{PSh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[d, "R\check{H}^0(S,-)"] \arrow[ur, Rightarrow] & D^+\,\mathrm{PSh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[d, "R\check{H}^0(X,-)"] \\
& D^+\,\mathrm{Ab} \arrow[r, equal] \arrow[d, "\mathrm{H}^1(-)"] \arrow[ur, Rightarrow] & D^+\,\mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] \\
& \mathrm{Ab} \arrow[r, equal] & \mathrm{Ab}
\end{tikzcd}$$

with natural transformations induced by the commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[d, "\iota_S"'] \arrow[dd, bend right=60, "\Gamma(\mathrm{S},\text{-})"'] & \mathrm{Sh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[l, "f_*"] \arrow[d, "\iota_X"] \arrow[dd, bend left=60, "\Gamma(\mathrm{X},\text{-})"] \\
\mathrm{PSh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[d, "\check{H}^0(S,-)"] & \mathrm{PSh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[l, "f_*"] \arrow[d, "\check{H}^0(X,-)"'] \\
\mathrm{Ab} \arrow[r, equal] & \mathrm{Ab}
\end{tikzcd} .$$

The isomorphisms

$$\check{H}^1(S,\iota_S(-)) \to \mathrm{H}^1_{et}(S,-) \text{ resp. } \check{H}^1(X,\iota_X(-)) \to \mathrm{H}^1_{et}(X,-)$$

in corollary 1.44 are given by the canonical natural transformation

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[d, "\iota_S"] \arrow[dr] & & & & \mathrm{Sh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[d, "\iota_X"] \arrow[dl] \\
\mathrm{PSh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[r, Rightarrow] \arrow[dr] & D^+\,\mathrm{Sh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[d, "R\iota_S"] & \text{resp.} & D^+\,\mathrm{Sh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[d, "R\iota_X"] & \mathrm{PSh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[l, Rightarrow] \arrow[dl] \\
& D^+\,\mathrm{PSh}(\mathrm{Et_S},\mathrm{Ab}) \arrow[d, "R\check{H}^0(S,-)"] & & D^+\,\mathrm{PSh}(\mathrm{Et_X},\mathrm{Ab}) \arrow[d, "R\check{H}^0(X,-)"] & \\
& D^+\,\mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] & & D^+\,\mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] & \\
& \mathrm{Ab} & & \mathrm{Ab} &
\end{tikzcd} .$$

We deduce the composition

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[d, "\iota_S"] \arrow[dr] & & \\
\mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, Rightarrow] \arrow[dr] & D^+ \mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f^*"] \arrow[d, "R\iota_S"] & D^+ \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "R\iota_X"] \\
& D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[d, "R\check{H}^0(S,-)"] \arrow[ur, Rightarrow] & D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "R\check{H}^0(X,-)"] \\
& D^+ \mathrm{Ab} \arrow[r, equal] \arrow[d, "\mathrm{H}^1(-)"] \arrow[ur, Rightarrow] & D^+ \mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] \\
& \mathrm{Ab} \arrow[r, equal] & \mathrm{Ab}
\end{tikzcd}$$

and the composition

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f^*"] \arrow[d, "\iota_S"] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "\iota_X"] \arrow[dr] & \\
\mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[ur, Rightarrow] \arrow[dr] & \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[r, Rightarrow] \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & D^+ \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "R\iota_X"] \\
& D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{S}}, \mathrm{Ab}) \arrow[r, "f_P^*"] \arrow[d, "R\check{H}^0(S,-)"] & D^+ \mathrm{PSh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab}) \arrow[d, "R\check{H}^0(X,-)"] \\
& D^+ \mathrm{Ab} \arrow[r, equal] \arrow[d, "\mathrm{H}^1(-)"] \arrow[ur, Rightarrow] & D^+ \mathrm{Ab} \arrow[d, "\mathrm{H}^1(-)"] \\
& \mathrm{Ab} \arrow[r, equal] & \mathrm{Ab}
\end{tikzcd}$$

to agree by using the compatibility results of propositions 6.5 and 6.6 By using the above considerations, we derive the commutativity of the diagram in question. □

6.2 Base change maps

For the rest of this section, we denote by

$$\begin{tikzcd}
X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
S' \arrow[r, "r1"] & S
\end{tikzcd} \tag{5}$$

a commutative diagram of schemes. Let $\mathcal{C}$ be either Set, Ab or ${}_{\Lambda}\mathrm{Mod}$ for Λ some commutative ring. We obtain a square of right adjoints

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C}) \arrow[d, "d1_*"] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}'}, \mathcal{C}) \arrow[l, "r2_*"'] \arrow[d, "d2_*"] \\
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C}) & \mathrm{Sh}(\mathrm{Et}_{\mathrm{S}'}, \mathcal{C}) \arrow[l, "r1_*"']
\end{tikzcd} .$$

There exist canonical isomorphisms

$$d1_* \circ r2_* \xrightarrow{\sim} r1_* \circ d2_*$$

and

$$r2^* \circ d1^* \xleftarrow{\sim} d2^* \circ r1^*$$

corresponding to the commutivity of the above diagram and to each other under bijection (4). Then, under bijection (3) resp. (2) those isomorphisms correspond to the same natural transformation

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C}) \arrow[r, "r2^*"] \arrow[d, "d1_*"] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}'}, \mathcal{C}) \arrow[d, "d2_*"] \\
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C}) \arrow[r, "r1^*"] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{S}'}, \mathcal{C})
\end{tikzcd} \tag{6}$$

which we call *base change map.* In the case of $\mathcal{C} = \mathrm{Ab}, {}_{\Lambda}\mathrm{Mod}$, we obtain also a derived version

$$\begin{tikzcd}
D^+\,\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C}) \arrow[r, "r2^*"] \arrow[d, "Rd2_*"] & D^+\,\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}'}, \mathcal{C}) \arrow[d, "Rd1_*"] \\
D^+\,\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C}) \arrow[r, "r1^*"] \arrow[ur, Rightarrow] & D^+\,\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}'}, \mathcal{C})
\end{tikzcd} \tag{7}$$

by remark (6.12) which we call *cohomological base change map* or also base change map if it is clear from the context which we are referring to.

Remark 6.15. Recall (7) be given by the composition

$$r1^* \circ Rd2_*(K^\bullet) \xrightarrow{\sim} r1^* \circ d2_*(I^\bullet) \to d1_* \circ r2^*(I^\bullet) \overset{d1_*(b)}{\to} d1_*(J^\bullet) \overset{\sim}{\leftarrow} Rd1_* \circ r2^*(K^\bullet)$$

for $K^\bullet$ a bounded below complex, $a : K^\bullet \to I^\bullet$ and $b : r2^* I^\bullet \to J^\bullet$ quasi-isomorphisms to complexes of injective sheaves and $r1^* \circ d2_*(I^\bullet) \to d1_* \circ r2^*(I^\bullet)$ induced by (6). If (6) is an isomorphism, then, the following are equivalent.

1. (7) yields an isomorphism.
2. $r2^*$ takes acyclic injective bounded below complexes to $d1_*$-acyclic complexes.

3. $r2^*$ takes injective sheaves to $d1_*$-acyclic objects.

Example 6.16. Consider a cartesian diagram of schemes

$$\begin{tikzcd}
X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
\mathrm{Spec}(k) \arrow[r, "r1"] & S
\end{tikzcd}$$

with

$$r1 : \mathrm{Spec}(k) \to S$$

a geometric point. Then,

$$\Gamma(\mathrm{Spec}(k), -) : \mathrm{Sh}(\mathrm{Et}_{\mathrm{Spec}(k)}, \mathcal{C}) \to \mathcal{C}$$

is an equivalence of categories. The base change map yields a natural transformation

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C}) \arrow[r, "r2^*"] \arrow[d, "d1_*"] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X'}}, \mathcal{C}) \arrow[d, "\Gamma(\mathrm{X'}\text{,-})"] \\
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C}) \arrow[r, "(\text{-})_{\bar{\mathrm{s}}}"] \arrow[ur, Rightarrow] & \mathcal{C}
\end{tikzcd}$$

using the compatibility of corollary 6.9. We convince ourselves that the counit associated to the left adjoint $(-)_{\bar{s}}$ is an isomorphism. A precise writing out of the definition shows the base change map pointwise

$$(d1_* \mathcal{F})_{\bar{s}} = \mathrm{colim}_{U \in \mathrm{NEt}_{\bar{s}}} \mathcal{F}(U \times_S X) \to r2^* \mathcal{F}(X')$$

to be induced by the family of units

$$\mathcal{F}(U \times_S X) \to f_* \circ f^*(\mathcal{F})(U \times_S X) \cong r2^* \mathcal{F}(X')$$

for each $U \in \mathrm{NEt}_{\bar{s}}$ and induced factorizations

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "r2"] & U \times_S X \arrow[dl, "g"] \\
X &
\end{tikzcd} .$$

Observe that we ommitted g^* since it is the restriction. In particular, for $\mathcal{C} = \mathrm{Ab}, {}_{\Lambda}\mathrm{Mod}$, the morphism on cohomology (7) yields the canonical morphism

$$(R^q d2_* \mathcal{F})_{\bar{s}} = \mathrm{colim}_{U \in \mathrm{NEt}_{\bar{s}}} \mathrm{H}^q_{et}(U \times_S X, g^* \mathcal{F}) \to \mathrm{H}^q_{et}(X', r2^* \mathcal{F})$$

induced by the family

$$\mathrm{H}^q_{et}(U \times_S X, g^* \mathcal{F}) \to \mathrm{H}^q_{et}(X', r2^* \mathcal{F})$$

of example 6.13.

Example 6.17. Consider the cartesian square of schemes

$$\begin{tikzcd}
X_0 \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
\mathrm{Spec}(k) \arrow[r, "r1"] & \mathrm{Spec}(A)
\end{tikzcd}$$

with (A, m) a strictly henselian local ring and $r1 = \bar{m}$ a geometric point at m. In that case, the trivial étale neighbourhood $\mathrm{Spec}(A)$ is initial in $\mathrm{NEt}_{\bar{m}}$. Combined with the previous example we identify the base change map resp. the cohomological base change map pointwise with

$$\Gamma(X, \mathcal{F}) \Rightarrow \Gamma(X_0, r2^* \mathcal{F}) \text{ resp. } \mathrm{H}^q(S, \mathcal{G}) \to \mathrm{H}^q(X, f^* \mathcal{G}) \tag{8}$$

of example 6.13.

Example 6.18. Consider a cartesian diagram of schemes

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
\mathrm{Spec}(\mathcal{O}_{S,\bar{s}}) \arrow[r, "r1"] & S
\end{tikzcd}$$

with $\mathcal{C} = \mathrm{Ab}, {}_\Lambda\mathrm{Mod}$, S and $d1$ quasi-compact and quasi-seperated and $\bar{s} : \mathrm{Spec}(k) \to S$ a geometric point. The base change map is

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_\mathrm{X}, \mathcal{C}) \arrow[r, "r2^*"] \arrow[d, "d1_*"] & \mathrm{Sh}(\mathrm{Et}_{X_{\bar{s}}}, \mathcal{C}) \arrow[d, "d2_*"] \\
\mathrm{Sh}(\mathrm{Et}_\mathrm{S}, \mathcal{C}) \arrow[r, "r1^*"] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{Spec}(\mathcal{O}_{S,\bar{s}})}, \mathcal{C})
\end{tikzcd} .$$

Observe

$$(-)_{\bar{s}} \circ d2_* \cong \Gamma(X_{\bar{s}}, -)$$

to be isomorphic by using corollary 3.24. Furthermore, $\mathrm{Spec}(\mathcal{O}_{S,\bar{s}})$ is the cofiltered limit of étale neighbourhoods of $\bar{s}$ which are in addition of finite presentation over S and affine by using that étale morphisms are locally given by such maps. We denote this full subcategory by $\mathrm{NEt}_{\bar{s}}^{aff,fp} \subset \mathrm{NEt}_{\bar{s}}$. Similar to example 6.16, we deduce the stalk at $\bar{s}$ of the base change map

$$(d1_* \mathcal{F})_{\bar{s}} = \mathrm{colim}_{U \in \mathrm{NEt}_{\bar{s}}^{aff,fp}} \mathcal{F}(U \times_S X) \to r2^* \mathcal{F}(X_{\bar{s}})$$

to be induced by the family of units of $f^* \dashv f_*$

$$\mathcal{F}(U \times_S X) \to f_* \circ f^*(\mathcal{F})(U \times_S X) = r2^* \mathcal{F}(X_{\bar{s}})$$

for each $U \in \mathrm{NEt}_{\bar{s}}^{aff,fp}$ and induced factorizations

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r, "f"] \arrow[d, "r2"] & U \times_S X \arrow[dl, "g"] \\
X &
\end{tikzcd}.$$

Furthermore, the cohomological base change map

$$(R^q d1_* \mathcal{F})_{\bar{s}} = \mathrm{colim}_{U \in \mathrm{NEt}_{\bar{s}}} \mathrm{H}^q_{et}(U \times_S X, g^* \mathcal{F}) \to \mathrm{H}^q_{et}(X_{\bar{s}}, r2^* \mathcal{F})$$

is induced by the family

$$\mathrm{H}^q_{et}(U \times_S X, g^* \mathcal{F}) \to \mathrm{H}^q_{et}(X_{\bar{s}}, r2^* \mathcal{F}).$$

6.3 Base change theorems

In this section we prove certain base change maps to be isomorphisms.

Example 6.19. Assume we are in the situation of example 6.18. Then, after composing with the stalk functor at $\bar{s}$ the base change map

$$(d2_* \mathcal{F})_{\bar{s}} \to r2^* \mathcal{F}(X_{\bar{s}})$$

and the cohomological base change map

$$(R^q d2_* \mathcal{F})_{\bar{s}} \to \mathrm{H}^q_{et}(X_{\bar{s}}, r2^* \mathcal{F})$$

are isomorphisms by the identification in example 6.18 and examples 2.53 and 2.59.

Example 6.20. Let

$$\begin{tikzcd}
X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
S' \arrow[r, "r1"] & S
\end{tikzcd}$$

be a cartesian diagram of schemes with $r1$ purely inseparable. Then, $r2$ is purely inseparable by remark 3.33 and its adjoint pairs of direct and inverse image functors define pairwise equivalences of categories by lemma 3.35. Thus, the base change map as well as the cohomologcial base change map are isomorphisms by corollary 6.10.

Example 6.21. Consider a cartesian diagram of schemes

$$\begin{tikzcd} V \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\ U \arrow[r, "r1"] & S \end{tikzcd}$$

with $r1$ (then $r2$) étale. In that case, $r1^*$ (and $r2^*$) are the canonical restrictions. The counit $r1^* \circ r1_* \Rightarrow 1$ is an isomorphism and the unit yields isomorphisms

$$\mathcal{F}(W) \to r2_* r2^* \mathcal{F}(W)$$

for every étale W/X factoring over $r2$. In particular, the unit of $r2^* \dashv r2_*$ is an isomorphism after applying $r1^* \circ d1_*$. We deduce the base change map to be an isomorphism by corollary 6.8. Assume $\mathcal{C}$ to be either Ab or ${}_\Lambda\operatorname{Mod}$. We check $r2^*$ to preserve flasque sheaves In particular, $r2^*$ turns injective sheaves into $d1_*$-acyclic sheaves. Combined with remark 6.15, the cohomological base change map is an isomorphism.

Theorem 6.22 (Proper base change theorem)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\ S' \arrow[r, "r1"] & S \end{tikzcd}$$

be a cartesian square of schemes with $d1$ proper. Then,

1. *the base change map (6)*

   $$r1^* \circ d1_* \Rightarrow d2_* \circ r2^*$$

   is an isomorphism for $\mathcal{C} = \operatorname{Set}$ (hence, for $\mathcal{C} = \operatorname{Ab}, {}_\Lambda\operatorname{Mod}$).

2. *the cohomological base change map (7)*

   $$r1^* \circ Rd1_* \Rightarrow Rd2_* \circ r2^*$$

   is an isomorphism for $\mathcal{C} = {}_\Lambda\operatorname{Mod}$ and $\Lambda = \mathbb{Z}/n$, $n > 0$.

3. the cohomological base change map (7)

$$r1^* \circ Rd1_*(K^\bullet) \to Rd2_* \circ r2^*(K^\bullet)$$

is an isomorphism for $K^\bullet$ a bounded below complex of torsion sheaves.

Remark 6.23. We observe 2. of the proper base change theorem to be a special case of 3. since every sheaf of $\mathbb{Z}/n$-modules for $n > 0$ is in particular a torsion abelian sheaf and the étale cohomology of some sheaf of $\mathbb{Z}/n$-modules agrees with its cohomology as an abelian sheaf. We will later prove that the converse is also true by using the compatibility of cohomology with filtered colimits.

Lemma 6.24. *We may reduce to complexes concentrated in degree 0 in 2. and 3. of the proper base change theorem. In particular, we need to prove the induced morphism*

$$r1^* \circ R^q d1_* \mathcal{F} \to R^q d2_* \circ r2^* \mathcal{F}$$

to be an isomorphism for every torsion sheaf $\mathcal{F}$ and $q \geq 0$.

Proof. By the previous remark, it suffices to prove the claim for 3. This is contained in the proof of [10, Tag 0DDE] □

6.4 Proof of the proper base change theorem

6.4.1 Summary

The proof of the proper base change theorem is based on a long sequence of reductions. In order to maintain an overview, we give a short summary here.

The first sequence of reductions mainly rests on the compatibility results given in proposition 6.5 and corollary 6.8. We reduce to

$$S \text{ and } S' \text{ being affine}$$

by choosing open affine coverings of S resp. S' and by using that the base change theorem is an isomorphism if $r1$ is étale. This allows us to apply the compatibility results with filtered colimits from section 2.8 since affine schemes are quasi-compact and quasi-separated. We reduce to

$$r1 \text{ being of finite type}$$

by writing $r1$ as a cofiltered limit of such morphisms. This allows us to reduce to proving the bijectivity on stalks at points closed in their fibre. We reduce to

$r1$ being a geometric point with induced field extension being a separable closure.

We may apply example 6.18 to reduce to

$r1$ being given by the quotient map at the closed point of a strictly henselian ring.

by using that $r1$ factorizes through its étale stalk. Then, the base change map is given as in example 6.17. By using that every torsion sheaf is a filtered colimit of sheaves of $\mathbb{Z}/n$-modules for $n > 0$, we apply theorem 2.43 to reduce from arbitrary torsion sheaves to $\mathbb{Z}/n$-sheaves. In particular,

2. of the proper base change theorem implies 3.

We give a proof of 1. of the proper base change theorem without further reduction. Furthermore, since every ring is a filtered colimit of $\mathbb{Z}$-algebras of finite type, in particular noetherian rings, we reduce to

S being in addition noetherian.

Then, we can apply remark 6.15 to

reduce from isomorphism to epimorphism in 2. of theorem 6.22.

We use Chow's Lemma to assume that

$d1$ is projective.

We reduce further to

X' having dimension lower or equal 1

by dropping the projectiveness assumption. Based on our considerations in section 2.5, we

reduce from arbitrary $\mathbb{Z}/n$-sheaves to the constant $\mathbb{Z}/n$-sheaf $\underline{\mathbb{Z}/n}$.

This is what we call the core case. To prove the core case we use our results on the cohomology of curves developed in chapter 5 and the results sketched in chapter 4.

6.4.2 Reductions

Fix a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\ S' \arrow[r, "r1"] & S \end{tikzcd}$$

with $d1$ proper. For the sake of clarity, we make a few conventions.

1. We will use the reduction from arbitrary complexes to complexes concentrated in degree zero of lemma 6.24 without further mentioning.

2. Observe forgetting the Λ-module structure

   $$\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, {}_{\Lambda}\mathrm{Mod}) \to \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathrm{Ab})$$

   to be exact and to preserve flasque sheaves. For $\Lambda = \mathbb{Z}/n$, the forgetful functor ${}_{\mathbb{Z}/n}\mathrm{Mod} \subset \mathrm{Ab}$ is in addition fully faithful. Therefore, given some sheaf $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_X, {}_{\mathbb{Z}/n}\mathrm{Mod}) \subset \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Ab})$, we will not distinguish its respective cohomology groups as a sheaf of $\mathbb{Z}/n$-modules and as a sheaf of abelian groups.

3. We will use that torsion sheaves are preserved under inverse image functors and that the base change of a proper morphism is again proper without further mentioning.

4. Occasionally we identify the inverse image functor with another functor along an equivalence of categories. In this case we use the compatibility of 6.9 without further mentioning.

Lemma 6.25. *In the situation of the proper base change theorem, we may reduce to S being affine.*

Proof. Let $\{U_i \subset S\}_{i \in I}$ be an affine open covering. Then, the induced family $\{U_i' = U_i \times_S S' \subset S'\}_{i \in I}$ is an étale covering. In particular, the base change map resp. cohomological base change map is an isomorphism iff it is an isomorphism restricted to every $\mathrm{Sh}(\mathrm{Et}_{U_i'}, \mathrm{Set})$ resp. $D^+\,\mathrm{Sh}(\mathrm{Et}_{U_i'}, \mathrm{Ab})$. We give only a proof for the base change map. We derive the claim for the cohomological base change map with almost identical arguments. We obtain a commutative cube

$$\begin{tikzcd}
V_i' \arrow[rr] \arrow[dr] \arrow[dd] & & V_i \arrow[dr] \arrow[dd] & \\
& X' \arrow[rr, "r2" description] \arrow[dd] & & X \arrow[dd, "d1" description] \\
U_i' \arrow[rr, "d2" description] \arrow[dr] & & U_i \arrow[dr] & \\
& S' \arrow[rr, "r1" description] & & S
\end{tikzcd}$$

with each face cartesian and each vertical arrow proper. This yields base change maps

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{V_i}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{V_i'}, \mathcal{C}) \arrow[d] & \mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{X'}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{V_i'}, \mathcal{C}) \arrow[d] \\
\mathrm{Sh}(\mathrm{Et}_S, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{U_i}, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{U_i'}, \mathcal{C}) & \mathrm{Sh}(\mathrm{Et}_S, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{S'}, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{U_i'}, \mathcal{C})
\end{tikzcd} .$$

Observe their respective compositions to agree by corollary 6.5. Then, all base change maps except

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{X'}, \mathcal{C}) \arrow[d] \\
\mathrm{Sh}(\mathrm{Et}_S, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{S'}, \mathcal{C})
\end{tikzcd}$$

are isomorphisms by example 6.21 and by assumption. Thus, the natural transformation

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{X'}, \mathcal{C}) \arrow[d] & \\
\mathrm{Sh}(\mathrm{Et}_S, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{S'}, \mathcal{C}) \arrow[r] & \mathrm{Sh}(\mathrm{Et}_{U_i'}, \mathcal{C})
\end{tikzcd}$$

with $\mathrm{Sh}(\mathrm{Et}_{S'}, \mathcal{C}) \to \mathrm{Sh}(\mathrm{Et}_{U_i'}, \mathcal{C})$ being the restriction is an isomorphism by corollary 6.8. This proves the claim. □

Lemma 6.26. *In the situation of the proper base change theorem, we may reduce to S and S' being affine.*

Proof. We may assume S to be affine by the previous lemma. Let $\{U_i \subset S'\}_{i \in I}$ be an affine open covering. The (cohomological) base change map is an isomorphism iff it is restricted to all Et_{U_i}. We obtain a commutative diagram

$$\begin{tikzcd}
V_i \arrow[r] \arrow[d] & X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
U_i \arrow[r] & S' \arrow[r, "r1"] & S
\end{tikzcd}$$

with both squares cartesian for every $i \in I$. The (cohomological) base change map corresponding to the left square is an isomorphism by example 6.21 since $U_i \to S'$ is étale. The (cohomological) base change map corresponding to the outer square is an isomorphism by assumption. Thus, the (cohomological) base change map is an isomorphism after postcomposing with $(U_i \to S')^*$ by corollary 6.8. □

Lemma 6.27. *In the situation of the proper base change theorem, we may reduce to* S, S' *being affine and* $r1$ *being of finite type.*

Proof. We may assume both $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ to be affine by the previous lemma. Then, we may apply the results of section 2.8 since affine schemes are quasi-compact and quasi-separated. Define $(f_i : B_i \to B)_I$ to be a filtered colimit cocone of A-algebras with each B_i being a finite type A-algebra. Then, $(F_i : S' \to S_i = \mathrm{Spec}(B_i))_{I^{op}}$ is a cofiltered limit cone. We obtain a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "G_i"'] \arrow[d, "d2"] \arrow[rr, bend left, "r2"] & X_i \arrow[r, "H_i"'] \arrow[d, "d_i"] & X \arrow[d, "d1"] \\
S' \arrow[r, "F_i"] \arrow[rr, bend right, "r1"'] & S_i \arrow[r, "L_i"] & S
\end{tikzcd}$$

with each square cartesian. The (cohomological) base change map induced by the right square is an isomorphism for all i by assumption. It suffices to prove that the (cohomological) base change map of the outer square is an isomorphism after applying $\Gamma(U, -)$ resp. $R\Gamma(U, -)$ for every $U \in \mathrm{Et}^{fp}_{S'}$. We will give a proof for the base change map.

There exists some $i_0 \in I$ and $U_{i_0} \in \mathrm{Et}^{fp}_{S_{i_0}}$ with

$$U \cong U_{i_0} \times_{S_{i_0}} S$$

by corollary 2.30. Let $\mathcal{F}$ be a sheaf of sets on Et_S. Define for $i \in I^{op}/i_0$

$$U_i = U_{i_0} \times_{S_{i_0}} S_i \text{ and } V_i = U_i \times_{S_i} X_i \text{ and } V = U \times_S X$$

to be the respective pullbacks. The units of the adjunctions $F_i^* \dashv F_{i*}$ resp. $G_i^* \dashv G_{i*}$ induce colimit cocones

$$(f_i(U_i) : L_i^* \circ d1_* \mathcal{F}(U_i) \to F_{i*} \circ F_i^* \circ L_i^* \circ d1_* \mathcal{F}(U_i) \cong r1^* \circ d1_* \mathcal{F}(U))_{i_0/I}$$

resp.

$$(g_i(V_i) : H_i^* \mathcal{F}(V_i) \to G_{i*} \circ G_i^* \circ H_i^* \mathcal{F}(V_i) \cong r2^* \mathcal{F}(V))_{i_0/I}$$

by example 2.53. Observe $f_i(U_i)$ resp. $g_i(V_i)$ to be the base change map corresponding to the lower resp. outer square of

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{X_i}, \mathcal{C}) \arrow[r, "G_i^*"] \arrow[d, "d_{i,*}"] \arrow[dd, bend right=60, "\Gamma(V_i,-)"'] & \mathrm{Sh}(\mathrm{Et}_{X'}, \mathcal{C}) \arrow[d, "d2_*"'] \arrow[dd, bend left=60, "\Gamma(V,-)"] \\
\mathrm{Sh}(\mathrm{Et}_{S_i}, \mathcal{C}) \arrow[r, "F_i^*"] \arrow[d, "\Gamma(U_i,-)"] & \mathrm{Sh}(\mathrm{Et}_{S'}, \mathcal{C}) \arrow[d, "\Gamma(U,-)"'] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

applied to $L_i^* \circ d1_* \mathcal{F}$ resp. $H_i^* \mathcal{F}$ by example 6.13. We consider the following diagram.

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \arrow[r, "H_i^*"] \arrow[d, "d1_*"] & \mathrm{Sh}(\mathrm{Et}_{X_i}, \mathcal{C}) \arrow[r, "G_i^*"] \arrow[d, "d_{i,*}"] & \mathrm{Sh}(\mathrm{Et}_{X'}, \mathcal{C}) \arrow[d, "d2_*"'] \\
\mathrm{Sh}(\mathrm{Et}_S, \mathcal{C}) \arrow[r, "L_i^*"] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{S_i}, \mathcal{C}) \arrow[r, "F_i^*"] \arrow[d, "\Gamma(U_i,-)"'] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{S'}, \mathcal{C}) \arrow[d, "\Gamma(U,-)"'] \\
 & \mathcal{C} \arrow[r, equal] \arrow[ur, Rightarrow] & \mathcal{C}
\end{tikzcd} \tag{9}$$

The base change map of the upper left square of (9) is an isomorphism by assumption. By using proposition 6.5 and proposition 6.6 applied to (9) we deduce the square

$$\begin{tikzcd}
L_i^* \circ d1_* \mathcal{F}(U_i) \arrow[r, "f_i(U_i)"] \arrow[d, "\cong"] & r1^* \circ d1_* \mathcal{F}(U) \arrow[d] \\
H_i^* \mathcal{F}(V_i) \arrow[r, "g_i(V_i)"] & r2^* \mathcal{F}(V)
\end{tikzcd}$$

where the left vertical morphism is induced by the base change map of the upper left square and the right vertical arrow is induced by the base change map in question to commute. Observe the left vertical morphism to be an isomorphism by assumption. We deduce the base change map in question

$$r1^* \circ d1_* \mathcal{F}(U) \to r2^* \mathcal{F}(V) \cong d2_* \circ r2^* \mathcal{F}(U)$$

to be an isomorphism since the maps $f_i(U_i)$ and $g_i(V_i)$ induce colimit cocones. This proves the claim for the base change map. We can prove the claim about the cohomological base change map with the same arguments replacing example 2.53 by example 2.59. □

Recall the following well known fact.

Lemma 6.28. *Let $f : Y \to \mathrm{Spec}(k)$ be a scheme of finite type over a field k. Then, the set of closed points is dense in Y.*

Corollary 6.29. *Let $f : Y \to T$ be a finite type morphim of schemes. A morphism $\mathcal{F} \to \mathcal{G}$ of étale sheaves of sets (hence of abelian groups or modules) on Et_Y is an isomorphism iff it is on stalks for every point $y \in Y$ closed in its fibre, i.e. closed in $Y \times_T \mathrm{Spec}(\kappa(f(y)))$.*

Proof. We only give a sketch of the proof leaving some claims to the reader. First, we observe the set of points of Y closed in their fibre to be dense in Y by using the previous lemma. Let U be an étale Y-scheme and s be an element of $\ker(\mathcal{F} \to \mathcal{G})(U)$. We need to prove that s becomes zero restricted to an étale covering of U. The image of s becomes zero at the stalks of a dense subset of U by assumption. Thus, there exists a family of étale morphisms $\{f_i : U_i \to U\}_{i \in I}$ such that s becomes zero in $\ker(\mathcal{F} \to \mathcal{G})(U_i)$ for every i and $\cup_I f_i(U_i) \subset U$ contains a dense subset. Since étale morphisms are open, $\cup_I f_i(U_i) \subset U$ is open. Hence, $\cup_I f_i(U_i) = U$ are equal and we deduce the claim. □

Lemma 6.30. *In the situation of the proper base change theorem, we may reduce to S being affine and $r1$ being given by a geometric point $r1 : \mathrm{Spec}(k) \to S$ with topological image $s \in S$ such that the induced $\kappa(s) \subset k$ is a separable closure.*

Proof. We may assume that $r1$ is a morphism of finite type of affine schemes by the previous reductions. Then, the base change map is an isomorphism iff it is on stalks for every point $s' \in S'$ closed in its fibre by the previous corollary. Let $s' \in S'$ be such a point. Choose a separable closure $\kappa(s') \subset k$. We obtain a commutative diagram

$$\begin{tikzcd}
X'_0 \arrow[r, "G_i"'] \arrow[d] & X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
\mathrm{Spec}(k) \arrow[r, "\bar{s}'"] & S' \arrow[r, "r1"] & S
\end{tikzcd}$$

with each square cartesian. The (cohomological) base change map corresponding to the left square is an isomorphism by assumption. If the (cohomological) base change map for the outer square is an isomorphism, then, the natural transformation

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{X'}}, \mathcal{C}) \arrow[d] & \\
\mathrm{Sh}(\mathrm{Et}_{\mathrm{S}}, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{S'}}, \mathcal{C}) \arrow[r, "(-)_{\bar{s}'}"] & \mathcal{C}
\end{tikzcd}$$

as well as its derived version are isomorphisms by corollary 6.8. This would prove the claim. The composition $\bar{s}' \circ r1$ factorizes through some geometric point $\bar{s} : \mathrm{Spec}(K) \to S$ with $\kappa(s) \subset K$ a separable closure. We obtain a commutative diagram

$$\begin{tikzcd}
X_0' \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X \arrow[d, "d1"] \\
\mathrm{Spec}(k) \arrow[r] \arrow[rr, bend right, "\bar{s}' \circ r1"'] & \mathrm{Spec}(K) \arrow[r, "\bar{s}"] & S
\end{tikzcd}$$

with each square cartesian. We need to prove the base change morphisms corresponding to the outer square to be isomorphisms. By assumption, the base change maps corresponding to the right square are isomorphisms. By corollary 6.8 it suffices to prove the base change maps corresponding to the left square to be isomorphisms. We deduce the field extension $\kappa(s) \subset \kappa(s')$ to be finite since s' is closed in $S' \times_S \mathrm{Spec}(\kappa(s))$ and $r1$ is of finite type. Thus, $K \subset k$ is purely inseparable. We deduce the claim by example 6.20. □

Lemma 6.31. *In the situation of the proper base change theorem, we may reduce to* $r1$ *being given by the quotient map* $\mathrm{Spec}(A/m) \to \mathrm{Spec}(A)$ *for some strictly henselian local ring* (A, m).

Proof. We may assume S to be affine and $r1$ to be given by a geometric point

$$r1 : \mathrm{Spec}(k) \to S$$

with topological image $s \in S$ such that the induced $\kappa(s) \subset k$ is a separable closure by the previous lemma. Denote by $\mathcal{O}_{S,r1}$ the étale stalk of S at $r1$. Then, $r1$ factorizes as

$$\mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{O}_{S,r1}) \to S$$

for the canonical morphism $\mathrm{Spec}(\mathcal{O}_{S,r1}) \to S$ such that $\mathcal{O}_{S,r1} \to k$ is (up to isomorphism) the quotient map at its closed point. We obtain a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "d2"] \arrow[rr, bend left, "r2"] & X_{r1} \arrow[r] \arrow[d] & X \arrow[d, "d1"] \\
\mathrm{Spec}(k) \arrow[r] \arrow[rr, bend right, "r1"'] & \mathrm{Spec}(\mathcal{O}_{S,r1}) \arrow[r] & S
\end{tikzcd}$$

with each square cartesian. By assumption, the base change maps corresponding to the left square are isomorphisms. By example 6.18 the base change map

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Et}_X, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{Et}_{X_{r1}}, \mathcal{C}) \arrow[d] & \\
\mathrm{Sh}(\mathrm{Et}_S, \mathcal{C}) \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Sh}(\mathrm{Et}_{\mathrm{Spec}(\mathcal{O}_{S,r1})}, \mathcal{C}) \arrow[r, "(-)_{r1}"] & \mathcal{C}
\end{tikzcd}$$

as well as its derived version are isomorphisms. We deduce the claim by corollary 6.8. □

Lemma 6.32. *2. of the proper base change theorem implies 3., i.e. we may restrict from general torsion sheaves to sheaves of $\mathbb{Z}/n$-modules with $n > 0$.*

Proof. We may assume that $r1$ is given by the quotient map

$$\mathrm{Spec}(A/m) \to \mathrm{Spec}(A)$$

for some strictly henselian local ring (A, m) by the previous lemma. Let $\mathcal{F}$ be an abelian torsion sheaf on Et_X. The base change map in question is given by

$$\mathrm{H}^q_{et}(X, \mathcal{F}) \to \mathrm{H}^q_{et}(X', r2^* \mathcal{F})$$

as in example 6.17. Define $\mathcal{F}_n = \ker(\mathcal{F} \xrightarrow{\cdot n} \mathcal{F}) \in \mathrm{Sh}(\mathrm{Et}_{X, \mathbb{Z}/n}\,\mathrm{Mod}) \subset \mathrm{Sh}(\mathrm{Et}_X, \mathrm{Ab})$ to be the kernel of pointwise multiplying with n for every $n > 0$. Then, the induced cocone $(\mathcal{F}_n \to \mathcal{F})_{\mathbb{N}}$ is a filtered colimit cocone. We obtain a compatible family of morphisms

$$\mathrm{H}^q_{et}(X, \mathcal{F}_n) \to \mathrm{H}^q_{et}(X', r2^* \mathcal{F}_n)$$

since the base change map is functorial. Those are isomorphisms by assumption. Observe X, X' to be quasi-compact and quasi-seperated, since they are proper over affine schemes. Thus, the induced cocones

$$(\mathrm{H}^q_{et}(X, \mathcal{F}_n) \to \mathrm{H}^q_{et}(X, \mathcal{F}))_{\mathbb{N}}$$

and

$$(\mathrm{H}^q_{et}(X', r2^* \mathcal{F}_n) \to \mathrm{H}^q_{et}(X', r2^* \mathcal{F}))_{\mathbb{N}}$$

are colimit cocones by theorem 2.43. Combined this proves the claim. □

We will give a reference for a proof of 1. in the proper base change theorem under the reductions made above. Thus, we will assume 1. of the proper base change theorem to be true for the rest of this section.

Corollary 6.33. *Let*

$$\begin{tikzcd} X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\ S' \arrow[r, "r1"] & S \end{tikzcd}$$

be a cartesian diagram of schemes with $d1$ *finite. Then, the cohomological base change map is an isomorphism.*

Proof. Observe $d1_*$ and $d2_*$ to be exact by theorem 3.30. Since the base change map is an isomorphism we deduce the claim. □

Lemma 6.34. *In 2. of the the proper base change theorem, we may reduce to* $r1$ *being given by the quotient map* $r1 : \mathrm{Spec}(A/m) \to \mathrm{Spec}(A)$ *for* (A, m) *the strict henselianization of a localization of some finite type* $\mathbb{Z}$*-algebra at some prime ideal. In particular,* A *is noetherian.*

Proof. We may assume $r1$ to be given by the quotient map

$$\mathrm{Spec}(A/m) \to \mathrm{Spec}(A)$$

for some strictly henselian local ring (A, m) by the previous lemma. We can prove that there exists a filtered colimit cocone

$$(f_i : C_j \to A)_I$$

with C_i strictly henselian and of the desired form by using the universal property of the strict henselianization. Denote by k_i the residue field of C_i at its closed point for each j. Then, the induced

$$(k_i \to k)_I$$

is also a colimit diagram. Observe $(\mathrm{Spec}(A) \to \mathrm{Spec}(C_i))_{I^{op}}$ to be a cofiltered limit cone. Then, there exists some proper morphism

$$d1_{i_0} : X_{i_0} \to \mathrm{Spec}(C_{i_0})$$

such that $d1_{i_0} \otimes_{C_{i_0}} A \cong d1$ by corollary 2.30. Define $X_i = X_{i_0} \otimes_{C_{i_0}} C_i$ to be the respective pullback for all $i \to i_0$. Then, the induced cones

$$(\phi_i : X \to X_i)_{I^{op}/i_0}$$

and

$$(\phi'_i : X' \to X_i \otimes_{C_i} k_i)_{I^{op}/i_0}$$

are cofilered limit cones. Denote by

$$r2_i : X_i \otimes_{C_i} k_i \to X_i$$

the induced morphism. Let $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{X},\mathbb{Z}/\mathrm{n}}\mathrm{Mod})$ be a sheaf. Recall $\mathrm{Sh}(\mathrm{Et}_{\mathrm{X},\mathbb{Z}/\mathrm{n}}\mathrm{Mod})$ to be locally finitely presentable since X is quasi-compact and quasi-separated. Thus, $\mathcal{F}$ is a filtered colimit of finitely presentable objects. With a similar argument as in the previous lemma, we may assume $\mathcal{F}$ to be finitely presentable. Without loss of generality there exists some $\mathcal{F}_{i_0} \in \mathrm{Sh}(\mathrm{Et}_{X_{i_0},\mathbb{Z}/n}\mathrm{Mod})$ finitely presentable such that $\phi_{i_0}^* \mathcal{F}_{i_0} \cong \mathcal{F}$ are isomorphic by theorem 2.31. Since the induced square

$$\begin{tikzcd}
X' \arrow[r, "r2"] \arrow[d, "\phi'_i"] & X \arrow[d, "\phi_i"] \\
X_i \otimes_{C_i} k_i \arrow[r, "r2_i"] & X_i
\end{tikzcd}$$

commutes, we obtain a commutative square

$$\begin{tikzcd}
\mathrm{H}^q_{et}(X_i, \mathcal{F}_i) \arrow[r] \arrow[d] & \mathrm{H}^q_{et}(X, \mathcal{F}) \arrow[d] \\
\mathrm{H}^q_{et}(X_i \otimes_{C_i} k_i, r2_i^* \mathcal{F}_i) \arrow[r] & \mathrm{H}^q_{et}(X', r2^* \mathcal{F})
\end{tikzcd}$$

with vertical arrows the base change maps associated to $r2_i$ resp. $r2$ and the horizontal arrows as in example 2.59 by functoriality. By assumption, the left vertical arrow is an isomorphism for all i. Furthermore, the horizontal maps induce colimit cocones by example 2.59. We deduce the first claim. At last, the strict henselianization of a noetherian ring remains noetherian by proposition 3.28. □

Remark 6.35. If 1. of the proper baseschange theorem holds, then, 2. is an isomorphism iff $r2^*$ takes injective sheaves to $d1_*$-acyclic objects by remark 6.15. Given an injective sheaf $I \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{X},\mathbb{Z}/\mathrm{n}}\mathrm{Mod})$ we deduce $r1^* \circ R^q d1_*(I) \cong 0$ to be zero for every $q > 0$. In particular, if 1. of the proper base change theorem holds, then, the cohomological base change map 2. of $\mathbb{Z}/n$-sheaves is an isomorphism iff

$$r1^* \circ R^q d1_*(I) \twoheadrightarrow R^q d2_* \circ r2^*(I)$$

is surjective for every injective sheaf I and $q > 0$.

Let us summarize the reductions regarding the cohomological base change morphism made so far.

Corollary 6.36. *Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes with (A, m, k) a strictly henselian local and noetherian ring. Denote by $i : X_0 \to X$ the fibre of f along the closed immersion $\mathrm{Spec}(A \to A/m)$. If the unit of the adjunction $i^* \dashv i_*$ induces an epimorphism*

$$\mathrm{H}^q_{et}(X, \mathcal{F}) \to \mathrm{H}^q_{et}(X_0, i^* \mathcal{F})$$

for all $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{X},\mathbb{Z}/\mathrm{n}}\mathrm{Mod})$ and $q, n > 0$, then, the proper base change theorem 6.22 holds.

Proof. Those are the previous reductions combined with the calculation of the cohomological base change map in example 6.17. □

A convenient technic to reduce from proper to projective morphisms is to use Chow's lemma, see [10, Tag 02O2]. Basically, this lemma allows us to replace proper by projective up to a surjective projective morphism.

Lemma 6.37. *Let $g : Y' \to Y$ be a morphism of schemes surjective at the level of topological spaces. Then, the unit μ of the adjunction $g^* \dashv g_*$ is pointwise a monomorphism.*

Proof. By the triangle identities, $g^*(\mu)$ is pointwise a (split) monomorphism. Every geometric point $\bar{y}$ at Y admits up to a field extension a lift $\bar{y}'$ to Y' since g is surjective. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_{\bar{y}} \arrow[r, "\mu_{\mathcal{F}}"] \arrow[d, equal] & (g_* g^* \mathcal{F})_{\bar{y}} \arrow[d, equal] \\
(g^* \mathcal{F})_{\bar{y}'} \arrow[r, "g^*(\mu_{\mathcal{F}})"] & (g^* g_* g^* \mathcal{F})_{\bar{y}'}
\end{tikzcd}$$

for every $\mathcal{F} \in \mathrm{Sh}(\mathrm{Et}_{Y,\mathbb{Z}/n}\mathrm{Mod})$. This proves $\mu_{\mathcal{F}}$ to be stalkwise a monomorphism. In particular, μ is pointwise a monomorphism. □

Lemma 6.38. *Let*

$$\begin{tikzcd}
Y' \arrow[r, "r3"] \arrow[d, "d4"] & Y \arrow[d, "d3"] \\
X' \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "d1"] \\
S' \arrow[r, "r1"] & S
\end{tikzcd}$$

be a commutative diagram of schemes with each square cartesian and $d1$ and $d3$ proper. Assume $d3$ to be surjective at the level of topological spaces. If the cohomological base change map for the outer and for the upper square is an isomorphism, then, it is for the lower square.

Proof. We need to prove that $r2^*$ turns injective $\mathbb{Z}/n$-sheaves into $d2_*$-acyclic sheaves by remark 6.35. Let $I \in \mathrm{Sh}(\mathrm{Et}_{\mathrm{X}}, {}_{\mathbb{Z}/\mathrm{n}}\,\mathrm{Mod})$ be injective. Choose an embedding

$$d3^* I \subset J$$

into an injective sheaf. Let μ be the unit of the adjunction $d3^* \dashv d3_*$. Then, the adjoint morphism $I \subset d3_* J$ of $d3^* I \subset J$ is a monomorphism since it is the composition of the morphism μ_I, which is a monomorphism by the previous lemma, and $d3_*(d3^* I \subset J)$. Since I is injective, $I \subset d3_* J$ is split. Therefore,

$$R^q d2_* \circ r2^*(I) \subset R^q d2_* \circ r2^*(d3_* J) \cong R^q d2_* \circ d4_*(r3^* J)$$

is split injective. Because the base change map is an isomorphism for each of the above squares, we obtain an isomorphism

$$R^q d2_* \circ r2^*(d3_* J) \cong R^q d2_* \circ d4_*(r3^* J).$$

On the other hand, Grothendieck's spectral sequence yields a converging spectral sequence

$$\mathrm{E}_2^{p,q} = R^p d2_* \circ R^q d4_*(r3^* J) \Rightarrow R^{p+q}(d2 \circ d4)_*(r3^* J) = \mathrm{E}^{p+q}\,.$$

Because cohomological base change is an isomorphism for the upper square, we obtain $R^q d4_*(r3^* J) \cong 0$ for all $q > 0$. We deduce $R^{p+q}(d2 \circ d4)_*(r3^* J) \cong 0$ for all $p+q > 0$ since the cohomological base change map is an isomorphism for the outer square. Combined, we obtain

$$R^q d2_* \circ d4_*(r3^* J) = 0$$

for all $q > 0$ and, hence, the claim. □

We can now invoke Chow's Lemma to reduce from proper to projective.

Corollary 6.39. *We may reduce in corollary 6.36 to f being projective.*

Proof. Chow's Lemma ([10, Tag 0200]) combined with remark [10, Tag 0201] yields a commutative diagram

$$\begin{tikzcd}
\mathbb{P}^n_A \arrow[dr] & X' \arrow[l, "i"'] \arrow[r, "\mu"] \arrow[d, "g"] & X \arrow[dl, "f"] \\
 & \operatorname{Spec}(A) &
\end{tikzcd}$$

with i an immersion, μ projective and surjective and $\mathbb{P}^n_A \to \operatorname{Spec}(A)$ the canonical morphism. Therefore, the composition $g = f \circ \mu$ is proper. Then, i is proper, hence, a closed immersion since it is proper and an immersion. We deduce g to be projective. In order to apply the previous lemma, we need to prove that corollary 6.36 implies 3. of the proper base change theorem if we replace proper by projective in both. We prove this by mindfully going through the above sequence of reductions and observe that we can replace proper by projective in any of them. Then, the base change morphisms corresponding to the upper and outer cartesian squares of the diagram

$$\begin{tikzcd}
Y' \arrow[r, "r3"] \arrow[d, "d4"] & X' \arrow[d, "\mu"] \\
Y \arrow[r, "r2"] \arrow[d, "d2"] & X \arrow[d, "f"] \\
\operatorname{Spec}(A/m) \arrow[r, "r1"] & \operatorname{Spec}(A)
\end{tikzcd}$$

are isomorphisms by our assumption. We apply lemma 6.38 to deduce the claim. □

Lemma 6.40. *We may reduce in corollary 6.36 to X_0 having dimension less or equal one.*

Proof. We may assume that f is projective by the previous lemma. In particular, we obtain a factorization

$$\begin{tikzcd}
X \arrow[r, hook, "p"] \arrow[d, "f"] & \mathbb{P}^n_A \arrow[dl] \\
\operatorname{Spec}(A) &
\end{tikzcd}$$

for some $n \in \mathbb{N}$ with p a closed immersion and $\mathbb{P}^n_A \to \operatorname{Spec}(A)$ the canonical morphism. We have already seen the base change maps to be isomorphisms for finite morphisms. Therefore, we may assume that f is the canonical morphism

$$\mathbb{P}^n_A \to \operatorname{Spec}(A)$$

since p is finite. By using the geometric argument of Lemma 7.3.8 in [2] we inductively reduce to $n = 1$ and deduce the claim. □

Remark 6.41. In the situation of lemma 6.40, the schemes X and X_0 are quasi-compact and quasi-separated. Since we may pass filtered colimits through cohomology groups by theorem 2.43 and $\mathrm{Sh}(\mathrm{Et}_{\mathrm{X},\mathbb{Z}/\mathrm{n}}\mathrm{Mod})$ is locally finitely presentable, we may restrict to $\mathcal{F}$ being finitely presentable.

Lemma 6.42. *We may reduce in corollary 6.36 to $dimX_0 \leq 1$ and $\mathcal{F} = \underline{\mathbb{Z}/n}$ for $n > 0$.*

Proof. We assume that X_0 has dimension at most one by the previous lemma. We prove that the base change map

$$\mathrm{H}^q_{et}(X, \mathcal{F}) \to \mathrm{H}^q_{et}(X_0, i^* \mathcal{F})$$

is an isomorphism for every $\mathcal{F}$ by induction over q. For $q = 0$ this is the sheaf of sets case. For general $q > 0$ we restrict to $\mathcal{F}$ being finitely presentable by the previous remark. By theorem 2.26, we can embed $\mathcal{F}$ into some $\oplus_{i=1}^n f_{i*}\underline{E_i}$ with E_i finite $\mathbb{Z}/n$-modules and $f_i : Y_i \to X$ finite morphisms of schemes. Furthermore, every $E_i \cong \oplus_{j=1}^{n_i} \mathbb{Z}/k_{ij}$ decomposes as a finite direct sum for some $k_{ij} > 0$. In particular,

$$\underline{E_i} \cong \oplus_{j=1}^{n_i} \underline{\mathbb{Z}/k_{ij}}$$

are isomorphic. We may assume without loss of generality $E_i = \mathbb{Z}/n_i$ since direct images commute with finite direct sums. We first prove base changing to induce an epimorphism

$$\mathrm{H}^p_{et}(X, \oplus_{i=1}^n f_{i*}\underline{\mathbb{Z}/n_i}) \to \mathrm{H}^p_{et}(X_0, i^* \oplus_{i=1}^n f_{i*}\underline{\mathbb{Z}/n_i})$$

for all $p > q$. Because cohomology as well as i^* commute with finite direct sums

$$\mathrm{H}^q_{et}(X, \oplus_{i=1}^n f_{i*}\underline{\mathbb{Z}/n_i}) \cong \oplus_{i=1}^n \mathrm{H}^q_{et}(X, f_{i*}\underline{\mathbb{Z}/n_i})$$

are natural isomorphic and we may assume $n = 1$. By using corollary 6.33 combined with the vertical version of corollary 6.8, we can show that it suffices to prove the base change map

$$\mathrm{H}^q_{et}(Y_1, \underline{\mathbb{Z}/n_1\mathbb{Z}}) \to \mathrm{H}^q_{et}(Y_{1,0}, i^*\underline{\mathbb{Z}/n_1\mathbb{Z}})$$

to be an epimorphism with $Y_{1,0} = Y_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$. Since f_1 is finite and $dimX_0 \leq 1$ we obtain $dimY_{1,0} \leq 1$. Thus, the above morphism is surjective for all q by assumption. We deduce the induced

$$\mathrm{H}^p_{et}(X, \oplus_{i=1}^n f_{i*}\underline{\mathbb{Z}/n_i}) \to \mathrm{H}^p_{et}(X_0, i^* \oplus_{i=1}^n f_{i*}\underline{\mathbb{Z}/n_i})$$

to be an isomorphism for all $p \leq q$ by induction hypothesis and to be an epimorphism for all $p \geq q+1$. Embed $\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}$ into an injective sheaf I. Denote by $\mathcal{G}$ the cokernel of the embedding. Observe

$$\mathrm{H}_{et}^{q}(X, I) = 0$$

to be zero for all $q > 0$ since I is injective. The long exact sequences in cohomology induce a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{H}_{et}^{q}(X,\mathcal{G}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_{et}^{q+1}(X,\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \arrow[r] \arrow[d, two heads] & 0 \arrow[d, hook] \\
0 \arrow[r] & \mathrm{H}_{et}^{q}(X_0,i^*\mathcal{G}) \arrow[r] & \mathrm{H}_{et}^{q+1}(X_0,i^*\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \arrow[r] & \mathrm{H}_{et}^{q+1}(X_0,i^*I)
\end{tikzcd}$$

with exact rows. Thus, the epimorphism

$$\mathrm{H}_{et}^{q+1}(X,\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \to \mathrm{H}_{et}^{q+1}(X_0,i^*\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i})$$

is a monomorphism by the four lemma and, hence, an isomorphism. Let

$$0 \to \mathcal{F} \subset \oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i} \twoheadrightarrow \mathcal{G}' \to 0$$

be a short exact sequence. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{et}^{q}(X,\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_{et}^{q}(X,\mathcal{G}') \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_{et}^{q+1}(X,\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}_{et}^{q+1}(X,\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \arrow[d, "\cong"] \\
\mathrm{H}_{et}^{q}(X_0,i^*\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) & \mathrm{H}_{et}^{q}(X_0,i^*\mathcal{G}') \arrow[r] & \mathrm{H}_{et}^{q+1}(X_0,i^*\mathcal{F}) \arrow[r] & \mathrm{H}_{et}^{q+1}(X_0,i^*\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i})
\end{tikzcd}$$

with exact rows from the long exact sequence in cohomology. We deduce

$$\mathrm{H}_{et}^{q+1}(X,\mathcal{F}) \to \mathrm{H}_{et}^{q+1}(X_0,i^*\mathcal{F})$$

to be a monomorphism by the four lemma. Write $\mathcal{G}'$ as a filtered colimit of finitely presentable objects. By using filtered colimits to be compatible with cohomology groups and to preserve both, monomorphisms and epimorphisms, we deduce

$$\mathrm{H}_{et}^{q+1}(X,\mathcal{G}') \to \mathrm{H}_{et}^{q+1}(X_0,i^*\mathcal{G}')$$

to be a monomorphism by using similar arguments. The four lemma applied to

$$\begin{tikzcd}
\mathrm{H}^q_{et}(X,\mathcal{G}') \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^{q+1}_{et}(X,\mathcal{F}) \arrow[r] \arrow[d, hook] & \mathrm{H}^{q+1}_{et}(X,\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^{q+1}_{et}(X,\mathcal{G}') \arrow[d, hook] \\
\mathrm{H}^q_{et}(X_0,i^*\mathcal{G}') \arrow[r] & \mathrm{H}^{q+1}_{et}(X_0,i^*\mathcal{F}) \arrow[r] & \mathrm{H}^{q+1}_{et}(X_0,i^*\oplus_{i=1}^{n} f_{i*}\underline{\mathbb{Z}/n_i}) \arrow[r] & \mathrm{H}^{q+1}_{et}(X_0,i^*\mathcal{G}')
\end{tikzcd}$$

proves $\mathrm{H}^{q+1}_{et}(X,\mathcal{F}) \to \mathrm{H}^{q+1}_{et}(X_0,i^*\mathcal{F})$ to be an isomorphism. □

6.4.3 Proof of the core case

By the sequence of reductions made above, the proper base change theorem holds iff the following theorem is true with 1. and 2. proved in this order.

Theorem 6.43. *Let* $f : X \to \mathrm{Spec}(A)$ *be a proper morphism with* (A,m,k) *a strictly henselian local ring. Denote by* $i : X_0 \to X$ *the fibre of* f *along the induced geometric point* $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ *and assume* $dim X_0 \leq 1$*. Then, the unit* $1 \Rightarrow i_*i^*$ *induces*

1. *isomorphisms*
$$\Gamma(X,\mathcal{F}) \to \Gamma(X_0,i^*\mathcal{F})$$
for all étale sheaves of sets.
2. *epimorphisms*
$$\mathrm{H}^q_{et}(X,\underline{\mathbb{Z}/n}) \to \mathrm{H}^q_{et}(X_0,i^*\underline{\mathbb{Z}/n})$$
for all $q,n > 0$*.*

For 2. we may in addition assume that A is noetherian.

The proof of 1. is very technical and does need the theory of spectral spaces, which we didn't develope. Therefore, we only give a reference.

Proof of 1. [10, Tag 0A3S] □

Proof of 2. By theorem 5.21, $\mathrm{H}^q_{et}(X_0,i^*\underline{\mathbb{Z}/n}) = 0$ for all $q > 3$. Thus, we only need to proof surjectivity for $q = 1,2$.
The case $q = 1$*:*
Notice $i^*\underline{\mathbb{Z}/n} \cong \underline{\mathbb{Z}/n}$ to be isomorphic by corollary 2.16. We obtain a commutatitive diagram

$$\begin{tikzcd}
\mathrm{H}^1_{et}(X,\underline{\mathbb{Z}/n}) \arrow[r, phantom, "\cong"] \arrow[d] & \mathrm{Tors}^{\cong}(\mathrm{Et}_X,\underline{\mathbb{Z}/n}) \arrow[d, "i^*"] \\
\mathrm{H}^1_{et}(X_0,\underline{\mathbb{Z}/n}) \arrow[r, phantom, "\cong"] & \mathrm{Tors}^{\cong}(\mathrm{Et}_{X_0},\underline{\mathbb{Z}/n})
\end{tikzcd}$$

as in remark 5.7 by the identification in theorem 5.3. Therefore, it suffices to lift any $\underline{\mathbb{Z}/n}$-torsor $\mathcal{F}$ on X_0 to one on X. By corollary 5.6, $\mathcal{F}$ is representable by a finite étale X_0-scheme U_0. In order to apply theorem 4.8, we write A as a filtered colimit of henselianizations of finite type $\mathbb{Z}$-algebras. Then, we may assume A to be the henselianization of some finite type $\mathbb{Z}$-algebra at some prime ideal by using similar arguments as in lemma 6.34. There exists up to isomorphism a unique étale and finite X-scheme U and an isomorphism $U \times_X X_0 \cong U_0$ by theorem 4.8. It remains to prove that h_U admits a $\underline{\mathbb{Z}/n}$-torsor structure such that the composition $\mathcal{F} \cong h_{U_0} \cong i^* h_U$ is an isomorphism of torsors. By Yoneda's Lemma, the action

$$\sigma_0 : \underline{\mathbb{Z}/n} \times h_{U_0} \to h_{U_0}$$

is induced by a morphism $\sqcup_{\mathbb{Z}/n} X_0 \times_{X_0} U_0 \to U_0$. Notice that both are finite étale X_0-schemes. Therefore, this morphism lifts uniquely to a morphism $\sqcup_{\mathbb{Z}/n} X \times_X U \to U$ by theorem 4.8. Hence, we obtain a group action

$$\sigma : \underline{\mathbb{Z}/n} \times h_U \to h_U$$

which is the pullback of the group action of $\mathcal{F}$. Observe σ to define a torsor iff it induces an isomorphism

$$(\sigma, pr_2) : \underline{\mathbb{Z}/n} \times h_U \to h_U \times h_U.$$

Being an isomorphism can be checked at the level of étale schemes over X by Yoneda's lemma. Since the morphism in question is the pullback of the isomorphism

$$(\sigma_0, pr_2) : \underline{\mathbb{Z}/n} \times h_{U_0} \to h_{U_0} \times h_{U_0}$$

it is an isomorphism by fully faithfulness in theorem 4.8.

The case $q = 2$:

Denote by p the characteristic of k. Write $n = p^l k$ with k not divided by p. Then, $\underline{\mathbb{Z}/n} \cong \underline{\mathbb{Z}/p^l} \oplus \underline{\mathbb{Z}/k\mathbb{Z}}$ are isomorphic. It suffices to treat the cases $n = k$ and $n = p^l$ separately since cohomology commutes with finite direct sums.

The case $n = p^l$:

There exists an increasing filtration

$$0 = (p^l)/(p^l) \subset \mathbb{Z}/p = (p^{l-1})/(p^l) \subset \cdots \subset (p^0)/(p^l) = \mathbb{Z}/p^l$$

with each cokernel isomorphic to $\mathbb{Z}/p^l$ as abelian groups. Thus, the constant sheaf $\underline{\mathbb{Z}/p^l}$

admits a finite filtration

$$0 = \underline{(p^l)/(p^l)} \subset \underline{\mathbb{Z}/p} = \underline{(p^{l-1})/(p^l)} \subset \cdots \subset \underline{(p^0)/(p^l)} = \underline{\mathbb{Z}/p^l}$$

with each quotient sheaf isomorphic to $\underline{\mathbb{Z}/p}$. Recall

$$\mathrm{H}^q_{et}(X_0, \underline{\mathbb{Z}/p}) \cong 0$$

to be zero for all $q > dimX_0 = 1$ by theorem 5.11. By induction on l combined with the long exact sequence in cohomology, we deduce the claim.

The case $n = k$:

Since n is invertible in $k = A/m$ it is invertible in A (as A is local hence $A^\times = A - m$). In particular, each scheme X, X_0 and $X_{0,red}$ is proper over some strictly henselian ring with n invertible in it. Observe $X_{0,red}$ to be noetherian with $dimX_{0,red}$ less or equal one since the topological spaces of X_0 and X agree. Then, after selecting a primitive n-th root of unity in A whose image is also a primitive n-root in A/m, we identify the constant sheaf $\underline{\mathbb{Z}/n}$ on X and X_0 with the sheaf of n-th roots of unity on the respective scheme by proposition 5.9. Let

$$j : X_{0,red} \to X_0$$

be the canonical morphism. The canonical morphisms

$$\mathcal{O}_{X,et} \to i_* \mathcal{O}_{X_0,et} \text{ and } \mathcal{O}_{X_0,et} \to j_* \mathcal{O}_{X_{0,red},et}$$

induce morphisms

$$R\Gamma(X, -)(\mathcal{O}^\times_{X,et}) \to R\Gamma(X_0, -)(\mathcal{O}^\times_{X_0,et})$$

and

$$R\Gamma(X_0, -)(\mathcal{O}^\times_{X_0,et}) \to R\Gamma(X_{0,red}, -)(\mathcal{O}^\times_{X_{0,red},et})$$

as in construction 2.17. The kummer sequence yields a commutative latter

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^1(X, \mathcal{O}^\times_{X,et}) \arrow[r] \arrow[d] & \mathrm{H}^2(X, \underline{\mathbb{Z}/n}) \arrow[r] \arrow[d, "\alpha"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(X_0, \mathcal{O}^\times_{X_0,et}) \arrow[r] \arrow[d] & \mathrm{H}^2(X_0, \underline{\mathbb{Z}/n}) \arrow[r] \arrow[d, "\beta"', "\cong"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(X_{0,red}, \mathcal{O}^\times_{X_{0,red},et}) \arrow[r, two heads] & \mathrm{H}^2(X_{0,red}, \underline{\mathbb{Z}/n}) \arrow[r] & \cdots
\end{tikzcd}$$

with lowest horizontal morphism being an epimorphism by corollary 5.24. We observe α

to be the base change map in question by example 6.13. By corollary 3.36 we deduce β to be an isomorphism. In particular, it suffices to prove both left vertical morphisms to be epimorphisms. Along the isomorphism in lemma 2.38 combined with the compatibility of remark 2.39 we need to prove the morphisms

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$$

and

$$\mathrm{Pic}(X_0) \to \mathrm{Pic}(X_{0,red})$$

induced by the base change of quasi-coherent sheaves to be surjective. The second morphism is surjective by corollary 2.42. Again, we reduce to A being the henselianization of some finite type $\mathbb{Z}$-algebra at a prime ideal in order to apply theorem 4.4. Define $X_n = X \otimes_A A/m^{n+1} \cong \underline{\mathrm{Spec}}_{X_{n+1}}(\mathcal{O}_{X_{n+1}}/m^n)$ and $\hat{X} = X \otimes_A \hat{A}$ with $\hat{A}$ the m-adic completion of A. In particular, the defining ideal of $X_n \to X_{n+1}$ is nilpotent. Applying corollary 2.42, each induced

$$\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$$

is surjective. Therefore, we can extend $\mathcal{L}_0 \in \mathrm{Pic}(X_0)$ to a compatible family $(\mathcal{L}_n)_{n\in\mathbb{N}}$ with $\mathcal{L}_n \in \mathrm{Pic}(X_n)$. By Grothendieck's existence theorem 4.6, there exists a coherent $\mathcal{O}_{\hat{X}}$-module $\hat{\mathcal{L}}$ and compatible isomorphisms $\hat{\mathcal{L}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{O}_{X_n} \cong \mathcal{L}_n$. We can prove $\hat{\mathcal{L}}$ to be an invertible sheaf on $\hat{X}$ since every $\mathcal{L}_n$ is an invertible sheaf on X_n. The functor

$$\mathrm{Pic}(- \otimes_A X) :_B \mathrm{Alg} \to \mathrm{Set}$$

is accessible by example 4.2. Then, there exists some $\tilde{\mathcal{L}} \in \mathrm{Pic}(X)$ such that the base change of $\tilde{\mathcal{L}}$ and $\hat{\mathcal{L}}$ become isomorphic in $\mathrm{Pic}(X_0)$ by theorem 4.4. Since the base change of $\hat{\mathcal{L}}$ is isomorphic to $\mathcal{L}$, we deduce the claim. $\square$

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